

Theory of Computer Science

A2. Mathematical Foundations

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Sets, Tuples, Relations

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 - **explicit**, listing all elements, e. g. $A = \{1, 2, 3\}$
 - **implicit**, specifying a **property** characterizing all elements, e. g. $A = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 3\}$
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German: Menge, Element, leere Menge, Mächtigkeit/Kardinalität

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German: Teilmenge, echte Teilmenge, Potenzmenge

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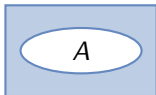
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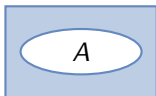
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German: Schnitt, Vereinigung, Differenz, Komplement

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 - $k = 2$: (ordered) pair
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German: k -Tupel, Komponente, Paar, Tripel

Cartesian Product

- for sets M_1, M_2, \dots, M_n , the **Cartesian product** $M_1 \times \dots \times M_n$ is the set $M_1 \times \dots \times M_n = \{\langle o_1, \dots, o_n \rangle \mid o_1 \in M_1, \dots, o_n \in M_n\}$.
- Example: $M_1 = \{a, b, c\}, M_2 = \{1, 2\}$,
 $M_1 \times M_2 = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$

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- special case: $M^k = M \times \dots \times M$ (k times)
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German: kartesisches Produkt

Relations

- an n -ary **relation** R over the sets M_1, \dots, M_n is a subset of their Cartesian product: $R \subseteq M_1 \times \dots \times M_n$.
- example with $M = \{1, 2\}$:
 $R_{\leq} \subseteq M^2$ as $R_{\leq} = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle\}$

German: (n -stellige) Relation

Functions

Functions

Definition (total function)

A (total) **function** $f : D \rightarrow C$ (with sets D, C)
maps **every value** of its **domain** D
to **exactly one value** of its **codomain** C .

German: (totale) Funktion, Definitionsbereich, Wertebereich

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- *square* : $\mathbb{Z} \rightarrow \mathbb{Z}$ with $\text{square}(x) = x^2$
- *add* : $\mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ with $\text{add}(x, y) = x + y$

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Example

- $square : \mathbb{Z} \rightarrow \mathbb{Z}$ with $square(x) = x^2$
- $add : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ with $add(x, y) = x + y$
- $add_{\mathbb{R}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $add_{\mathbb{R}}(x, y) = x + y$

Functions: Example

Example

Let $Z = \{z_0, z_1, z_2, z_e\}$ and $\Gamma = \{0, 1, \square\}$.

Define $\delta : Z \setminus \{z_e\} \times \Gamma \rightarrow Z \times \Gamma \times \{L, R, N\}$ by

δ	0	1	\square
z_0	$\langle z_0, 0, R \rangle$	$\langle z_0, 1, R \rangle$	$\langle z_1, \square, L \rangle$
z_1	$\langle z_2, 1, L \rangle$	$\langle z_1, 0, L \rangle$	$\langle z_e, 1, N \rangle$
z_2	$\langle z_2, 0, L \rangle$	$\langle z_2, 1, L \rangle$	$\langle z_e, \square, R \rangle$

Then, e. g., $\delta(z_0, 1) = \langle z_0, 1, R \rangle$

Partial Functions

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A **partial function** $f : X \rightarrow_p Y$ maps every value in X to **at most** one value in Y .

If f does not map $x \in X$ to any value in Y , then f is **undefined** for x .

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Example

$f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow_p \mathbb{N}_0$ with

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$

Summary

Summary

- **sets:** unordered, contain every element at most once
- **tuples:** ordered, can contain the same object multiple times
- **Cartesian product:** $M_1 \times \cdots \times M_n$ set of all n -tuples where the i -th component is in M_i
- **function** $f : X \rightarrow Y$ maps every value in X to exactly one value in Y
- **partial function** $g : X \rightarrow_p Y$ may be undefined for some values in X