Understanding the Search Behaviour of Greedy Best-First Search

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Introduction
Open Questions

- Which states is GBFS *guaranteed* to expand?
- Which states is GBFS guaranteed *not* to expand?
- Which states may GBFS *potentially* expand?

**Note:** Partly answered for A* (based on $f$-value) and for GBFS (based on high-water mark).
State Space Topology

- **state space**: generative model with initial state, goal states and successor function
- **heuristic**: assigns non-negative values to states
- **state space topology**: state space + heuristic
State Space Topology

Example

- $h = 6$: X
- $h = 5$: B, A
- $h = 4$: E, D
- $h = 3$: G, F, C
- $h = 2$: H
- $h = 1$: M, L, U, N, S, Q
- $h = 0$: Z, T, R, P, J, K, Y
Greedy Best-First Search

- expansion: generates successors of a state
- greedy best-first search: iteratively expands states with lowest heuristic value
- tie-breaking: selects a state among states with equal heuristic values
Greedy Best-First Search

Example

\[ h = 6 \]

\[ h = 5 \]

\[ h = 4 \]

\[ h = 3 \]

\[ h = 2 \]

\[ h = 1 \]

\[ h = 0 \]
Greedy Best-First Search

Example

\[
\begin{align*}
 & h = 6 \quad \times \\
 & h = 5 \\
 & h = 4 \\
 & h = 3 \\
 & h = 2 \\
 & h = 1 \\
 & h = 0
\end{align*}
\]
Greedy Best-First Search

Example

\[
\begin{align*}
    h &= 6 & X \\
    h &= 5 \\
    h &= 4 \\
    h &= 3 \\
    h &= 2 \\
    h &= 1 \\
    h &= 0
\end{align*}
\]
Greedy Best-First Search

Example

$h = 6$  \(X\)

$h = 5$

$h = 4$

$h = 3$

$h = 2$

$h = 1$

$h = 0$
Greedy Best-First Search

Example

\[ h = 6 \quad \text{X} \]

\[ h = 5 \]

\[ h = 4 \]

\[ h = 3 \]

\[ h = 2 \]

\[ h = 1 \]

\[ h = 0 \]
Greedy Best-First Search

Example

- $h = 6$: X
- $h = 5$: B
- $h = 4$: C
- $h = 3$: K, J
- $h = 2$: P
- $h = 1$: 
- $h = 0$: 

Understandig the Search Behaviour of Greedy Best-First Search
Greedy Best-First Search

Example

\[
\begin{align*}
  h &= 6 \quad X \\
  h &= 5 \\
  h &= 4 \\
  h &= 3 \quad K, J, I \\
  h &= 2 \\
  h &= 1 \\
  h &= 0
\end{align*}
\]
Greedy Best-First Search

Example

\[ h = 6 \quad \text{X} \]

\[ h = 5 \]

\[ h = 4 \]

\[ h = 3 \]

\[ h = 2 \]

\[ h = 1 \]

\[ h = 0 \]
Greedy Best-First Search

Example

- $h = 6$: X
- $h = 5$: B, A
- $h = 4$: G, F, C
- $h = 3$: K, J, I
- $h = 2$: U
- $h = 1$: P, N
- $h = 0$: T
High-Water Marks
High-Water Marks

Definition (high-water mark)

The high-water mark is the largest heuristic value of a state that GBFS starting from a state (or a set of states) must expand before reaching a goal state.
Example

high-water mark of state $P$: 4
High-Water Mark of State Set

Example

High-water mark of state set \{J, P\}: 3
Theorem (Wilt & Ruml, SoCS 2014)

GBFS is guaranteed to not expand a state whose heuristic value is larger than high-water mark of initial state.
Earlier Result

**Example**

never expanded states: \{X\}
Benches
Bench Exit States

Definition (bench exit state)

Bench exit state is a state which has a successor that has lower high-water mark or that is a goal state.
Bench Exit States

Example

\[
h = 6 \quad X
\]

\[
h = 5
\]

\[
h = 4
\]

\[
h = 3
\]

\[
h = 2
\]

\[
h = 1
\]

\[
h = 0
\]

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**Bench Exit States**

**Example**

- **$h = 6$**: Node $X$
- **$h = 5$**: Node $A$
- **$h = 4$**: Nodes $B$, $C$, $D$
- **$h = 3$**: Nodes $E$, $F$, $G$, $H$
- **$h = 2$**: Nodes $J$, $K$, $L$, $M$
- **$h = 1$**: Nodes $N$, $P$, $Q$
- **$h = 0$**: Nodes $R$, $S$, $T$, $U$, $V$, $W$, $X$, $Y$, $Z$
Bench Exit Property

Theorem (bench exit property)

Whenever GBFS expands a bench exit state, all previously generated states will never be expanded for the rest of the algorithm run.

Note: GBFS makes progress when bench exit state is expanded.
Bench Exit Property

Example

\[ h = 6 \]
\[ h = 5 \]
\[ h = 4 \]
\[ h = 3 \]
\[ h = 2 \]
\[ h = 1 \]
\[ h = 0 \]
Bench Exit Property

Example

\[ h = 6 \quad \text{X} \quad h = 5 \]

\[ h = 4 \]

\[ h = 3 \]

\[ h = 2 \]

\[ h = 1 \]

\[ h = 0 \]
## Bench Exit Property

**Example**

<table>
<thead>
<tr>
<th>$h$</th>
<th>X</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>C</td>
<td></td>
<td>D</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Bench Exit Property

Example

\[ h = 6 \quad X \]
\[ h = 5 \]
\[ h = 4 \]
\[ h = 3 \]
\[ h = 2 \]
\[ h = 1 \]
\[ h = 0 \]
Bench Exit Property

Example

- $h = 6$  
- $h = 5$  
- $h = 4$  
- $h = 3$  
- $h = 2$  
- $h = 1$  
- $h = 0$
Bench Exit Property

Example

\[ h = 6 \quad X \]

\[ h = 5 \]

\[ h = 4 \]

\[ h = 3 \]

\[ h = 2 \]

\[ h = 1 \]

\[ h = 0 \]
Bench Exit Property

Example

\begin{itemize}
  \item $h = 6$
  \item $h = 5$
  \item $h = 4$
  \item $h = 3$
  \item $h = 2$
  \item $h = 1$
  \item $h = 0$
\end{itemize}
Bench Exit Property

Example

\[ h = 6 \]
\[ h = 5 \]
\[ h = 4 \]
\[ h = 3 \]
\[ h = 2 \]
\[ h = 1 \]
\[ h = 0 \]
Bench Exit Property

Example

\[ h = 6 \]
\[ h = 5 \]
\[ h = 4 \]
\[ h = 3 \]
\[ h = 2 \]
\[ h = 1 \]
\[ h = 0 \]
Bench Exit Property

Example

\[ h = 6 \]

\[ h = 5 \]

\[ h = 4 \]

\[ h = 3 \]

\[ h = 2 \]

\[ h = 1 \]

\[ h = 0 \]
Bench Exit Property

Example

$h = 6$

$h = 5$

$h = 4$

$h = 3$

$h = 2$

$h = 1$

$h = 0$
Bench Exit Property

Example

\[
\begin{align*}
h &= 6 \\
h &= 5 \\
h &= 4 \\
h &= 3 \\
h &= 2 \\
h &= 1 \\
h &= 0
\end{align*}
\]
Bench Exit Property

Example

\[
\begin{align*}
 h &= 6 \\
 h &= 5 \\
 h &= 4 \\
 h &= 3 \\
 h &= 2 \quad &\text{U} \\
 h &= 1 \\
 h &= 0 \quad &\text{T} \\
\end{align*}
\]
### Example

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$h = 6$</td>
</tr>
<tr>
<td>5</td>
<td>$h = 5$</td>
</tr>
<tr>
<td>4</td>
<td>$h = 4$</td>
</tr>
<tr>
<td>3</td>
<td>$h = 3$</td>
</tr>
<tr>
<td>2</td>
<td>$h = 2$</td>
</tr>
<tr>
<td>1</td>
<td>$h = 1$</td>
</tr>
<tr>
<td>0</td>
<td>$h = 0$</td>
</tr>
</tbody>
</table>
Benches

Definition (bench)

A bench contains all states that GBFS starting with a given set of states can expand until expansion of a bench exit state. It is empty if the given set of states contains a goal state. It is associated with high-water mark of the given set of states.
Example

states on bench defined by \( \{ J, P \} : \{ I, J, P, K \} \)
Bench Transition Systems

Definition (bench transition system)

A bench transition system contains all benches which are reachable from the bench that starts with the initial state.

A successor bench is defined by the successor states of a bench exit state.
Bench Transition Systems

Example

$h = 5$

$h = 4$

$h = 3$

$h = 2$

$h = 1$

$h = 0
Results

**Theorem**

GBFS potentially expands a state if it is on at least one bench from bench transition system.

**Theorem**

GBFS is guaranteed to not expand a state that is not on a bench of the bench transition system.
Results

Example

never expanded states: \( \{B, E, H, T, U, X, Y, Z\} \)
Craters
**Surfaces**

**Definition (surface)**

A state is on the *surface* of a *bench* if its *heuristic* value is the *high-water mark* of the *bench*.

*Note:* Is often called *heuristic plateau* or *uninformed heuristic region*. 
Surfaces

Example

$h = 5$

$h = 4$

$h = 3$

$h = 2$

$h = 1$

$h = 0$
**Crater Entry States**

**Definition (crater entry state)**

A crater entry state is a state that is on the surface of a bench and that has a successor which is on a bench but not on a surface.
Crater Entry States

Example

\[h = 5\]

\[h = 4\]

\[h = 3\]

\[h = 2\]

\[h = 1\]

\[h = 0\]
Craters

Definition (crater)

A crater contains all states that GBFS starting with a given crater entry state expands until expansion of a state from the surface.

Note: Is often called local minimum or uninformed heuristic region.
Craters

Example

$h = 5$

$h = 4$

$h = 3$

$h = 2$

$h = 1$

$h = 0$
Theorem

Whenever GBFS expands a crater entry state $s$, then GBFS is guaranteed to expand all states in the crater defined by $s$. 
Conclusion
Conclusion

- exact characterization of potentially expanded and never expanded states
- characterization of surely expanded states given some conditions
- better understanding of search behaviour and search progress