

Bounded Intention Planning Revisited: Proof

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We claim that each applicable operator partition induces a strong semistubborn set such that the applicable operators are the same as the operators in the partition.

Theorem 1. *Let s be a state, $X \in P_s$ be an applicable partition. Then $T_s := X \cup \{o \mid o \text{ interferes with } o' \in X\}$ is a strong semistubborn set with the same applicable operators as X .*

We split the proof for Theorem 1 into three parts, proving the claim for each possible type of operator partitions separately.

We will use the following notation: an operator $o \in \bar{\mathcal{O}}$ affects a variable $v \in \bar{\mathcal{V}}$ if and only if $v \in \text{vars}(\text{eff}_o)$ (and hence also $v \in \text{vars}(\text{pre}_o)$).

Proposition 1. *First case of Theorem 1. Let s be a state, $X \in P_s$ be an applicable operator partition of type Fire_o , for the original operator $o \in \mathcal{O}$. Then the set $T_s := X \cup \{o' \mid o' \text{ interferes with } o'' \in X\}$ is a strong semistubborn set with the same applicable operators as X .*

Proof. We first note that Fire_o contains exactly one operator, namely $\text{Fire}(o)$. Wlog. we assume that o affects $v \in \mathcal{V}$, i. e. $v \in \text{vars}(\text{pre}_o)$ and $v \in \text{vars}(\text{eff}_o)$. Furthermore, o possibly has a prevail-condition on some variable $w \in \mathcal{V}$, $w \neq v$, i. e. $w \in \text{vars}(\text{prv}_o)$. By definition $\text{Fire}(o)$ has the following properties:

$$\begin{aligned} \text{pre}_{\text{Fire}(o)}[v] &= \text{pre}_o[v] \\ \text{eff}_{\text{Fire}(o)}[v] &= \text{eff}_o[v] \\ \text{pre}_{\text{Fire}(o)}[O_v] &= o \\ \text{eff}_{\text{Fire}(o)}[O_v] &= \text{free} \\ \text{prv}_{\text{Fire}(o)}[w] &= \text{prv}_o[w] \quad \forall w \in \text{vars}(\text{prv}_o) \\ \text{pre}_{\text{Fire}(o)}[O_w] &= \text{frozen} \quad \forall w \in \text{vars}(\text{prv}_o) \\ \text{eff}_{\text{Fire}(o)}[O_w] &= \text{free} \quad \forall w \in \text{vars}(\text{prv}_o) \\ \text{pre}_{\text{Fire}(o)}[C_w] &= v \quad \forall w \in \text{vars}(\text{prv}_o) \\ \text{eff}_{\text{Fire}(o)}[C_w] &= \text{free} \quad \forall w \in \text{vars}(\text{prv}_o) \end{aligned}$$

Second, note that we know the following about state s , considering that $\text{Fire}(o)$ is applicable in s :

$$\begin{aligned} s[v] &= \text{pre}_o[v] \\ s[w] &= \text{prv}_o[w] \quad \forall w \in \text{vars}(\text{prv}_o) \\ s[O_v] &= o \\ s[O_w] &= \text{frozen} \quad \forall w \in \text{vars}(\text{prv}_o) \\ s[C_w] &= v \quad \forall w \in \text{vars}(\text{prv}_o) \end{aligned}$$

We show that all operators o' that interfere with $\text{Fire}(o)$ are not applicable in s . Thus $\text{Fire}(o)$ is the only applicable operator in T_s . Second, we show that for all these operators $o' \in T_s$ (except for $\text{Fire}(o)$), T_s already contains a necessary enabling set for o' in s .

Let $u \neq \text{Fire}(o)$ be an arbitrary operator interfering with $\text{Fire}(o)$. Whenever we mention o' in the following, we refer to an operator $o' \in \mathcal{O}$, $o' \neq o$.

1. If u disables $\text{Fire}(o)$, we must distinguish the following cases.

(a) $v \in \text{vars}(\text{eff}_u)$ and $\text{eff}_u[v] \neq \text{pre}_{\text{Fire}(o)}[v]$. By definition, only fire operators can affect variables in \mathcal{V} . Let $u = \text{Fire}(o')$. From the definition of augmented operators, a fire operator affecting variable v also affects O_v . We conclude $\text{pre}_{\text{Fire}(o')}[O_v] = o' \neq o = s[O_v]$. Therefore $\text{Fire}(o')$ is not applicable in s .

$\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{Fire}(o')$ in s : only $\text{Fire}(o)$ can change O_v from o to free , which is required because all SetO operators (which can set O_v to o') require $O_v = \text{free}$ as precondition.

(b) $O_v \in \text{vars}(\text{eff}_u)$ and $\text{eff}_u[O_v] \neq \text{pre}_{\text{Fire}(o)}[O_v]$. We have to distinguish three possible types for u :

i. $u = \text{Fire}(o')$. If $v \in \text{vars}(\text{eff}_{o'})$, Case 1(a) applies. If $v \in \text{vars}(\text{prv}_{o'})$, we have $\text{pre}_{\text{Fire}(o')}[O_v] = \text{frozen} \neq o = s[O_v]$. Thus $\text{Fire}(o')$ is not applicable. $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{Fire}(o')$ for the same reasons as in Case 1(a).

ii. $u = \text{SetO}(o')$. We have $\text{pre}_{\text{SetO}(o')}[O_v] = \text{free} \neq o = s[O_v]$ and Case 1(a) applies.

iii. $u = \text{Freeze}(v, x)$ for $x \in \mathcal{D}(v)$. We have $\text{pre}_{\text{Freeze}(v, x)}[O_v] = \text{free} \neq o = s[O_v]$ and Case 1(a) applies.

(c) $w \in \text{vars}(\text{eff}_u)$ for variable $w \in \text{vars}(\text{prv}_{\text{Fire}(o)})$ and $\text{eff}_u[w] \neq \text{prv}_{\text{Fire}(o)}[w]$. Only fire operators can affect original variables from \mathcal{V} . Let $u = \text{Fire}(o')$. We conclude $\text{pre}_{\text{Fire}(o')}[O_w] = o' \neq \text{frozen} = s[O_w]$. Therefore $\text{Fire}(o')$ is not applicable.

We claim that $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{Fire}(o')$ in s : we observe that in order to apply $\text{Fire}(o')$, O_w must not have the value frozen . Consider an operator \tilde{o} that changes the value of O_w from frozen to free . Note that only a fire operator $\tilde{o} := \text{Fire}(o'')$ with $w \in \text{vars}(\text{prv}_{o''})$ can achieve this, because exactly for such fire operators, we have $\text{pre}_{\text{Fire}(o'')}[O_w] = \text{frozen}$ and $\text{eff}_{\text{Fire}(o'')}[O_w] = \text{free}$. If $v \in \text{vars}(\text{eff}_{\text{Fire}(o'')})$, then $\text{pre}_{\text{Fire}(o'')}[O_v] = o'' \neq o = s[O_v]$ and thus $\text{Fire}(o)$ must be applied first, as argued in Case 1(a). If $v \notin \text{vars}(\text{eff}_{\tilde{o}})$, then for some variable $v', v' \in \text{vars}(\text{eff}_{\text{Fire}(o'')})$. By definition of $\text{Fire}(o'')$, $\text{pre}_{\text{Fire}(o'')}[C_w] = v' \neq v = s[C_w]$. All operators that set C_w from v to free also affect v and thus have a precondition on O_v , and thus $\text{Fire}(o)$ needs to be applied before them.

(d) $O_w \in \text{vars}(\text{eff}_u)$ for variable $w \in \text{vars}(\text{prv}_{\text{Fire}(o)})$ and $\text{eff}_u[O_w] \neq \text{pre}_{\text{Fire}(o)}[O_w]$. We have to distinguish two possible types for u (freeze operators for variable w do not disable $\text{Fire}(o)$):

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- i. $u = \text{Fire}(o')$. If $w \in \text{vars}(\text{eff}_{o'})$, we have $w \in \text{vars}(\text{eff}_{\text{Fire}(o')})$ and Case 1(c) applies. If $w \in \text{vars}(\text{prv}_{o'})$, there must be a variable $v' \in \mathcal{V}$ for which $v' \in \text{vars}(\text{eff}_{o'})$. If $v' = v$, then $\text{pre}_{\text{Fire}(o')}[O_v] = o' \neq o = s[O_v]$ and $\text{Fire}(o')$ is not applicable in s . $\{\text{Fire}(o)\}$ is a necessary enabling set in s for $\text{Fire}(o')$ as argued in Case 1(a). If $v' \neq v$, then $\text{pre}_{\text{Fire}(o')}[C_w] = v' \neq v = s[C_w]$ and thus $\text{Fire}(o')$ is not applicable in s . $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{Fire}(o')$, because all operators that set C_w from v to free also affect v and thus have a precondition on O_v , and thus $\text{Fire}(o)$ needs to be applied before them, because $s[O_v] = o$.
 - ii. $u = \text{SetO}(o')$. We have $\text{pre}_{\text{SetO}(o')}[O_w] = \text{free} \neq \text{frozen} = s[O_w]$ and thus $\text{SetO}(o')$ is not applicable in s . $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{SetO}(o')$: Case 1(c) applies.
- (e) $C_w \in \text{vars}(\text{eff}_u)$ for variable $w \in \text{vars}(\text{prv}_{\text{Fire}(o)})$ and $\text{eff}_u[C_w] \neq \text{pre}_{\text{Fire}(o)}[C_w]$. We have to distinguish two possible types for u :
- i. $u = \text{Fire}(o')$. We know that $\text{pre}_{\text{Fire}(o')}[C_w] = v'$ for some variable $v' \in \mathcal{V}$. The case reduces to Case 1(d)i, the case “If $w \in \text{vars}(\text{prv}_{o'})$ ”.
 - ii. $u = \text{SetC}(w, c)$ for some $c \in \text{CG}(w)$. We have $\text{pre}_{\text{SetC}(w, c)}[C_w] = \text{free} \neq v = s[C_w]$ and thus $\text{SetC}(w, c)$ is not applicable in s . $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{SetC}(w, c)$, because only fire operators affecting v can change C_w from v to free and $s[O_v] = o$ (see also Case 1(d)i, second case “If $v' \neq v$ ”).

2. If $\text{Fire}(o)$ disables u , we must distinguish the following cases.

- (a) $v \in \text{vars}(\text{pre}_u)$ and $\text{pre}_u[v] \neq \text{eff}_{\text{Fire}(o)}[v]$. Only fire operators $u = \text{Fire}(o')$ can have original variables from \mathcal{V} as a precondition. From the definition of fire operators, we conclude $\text{pre}_{\text{Fire}(o')}[O_v] = o' \neq o = s[O_v]$. Hence $\text{Fire}(o')$ is not applicable in s . $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{Fire}(o')$ because only $\text{Fire}(o)$ can change O_v from o to free , which is precondition for all SetO operators, which are in turn needed to set O_v to o' (see also Case 1(a)).
- (b) $v \in \text{vars}(\text{prv}_u)$ and $\text{prv}_u[v] \neq \text{eff}_{\text{Fire}(o)}[v]$. We must distinguish three possible types for u :
 - i. $u = \text{Fire}(o')$. From $v \in \text{vars}(\text{prv}_{o'})$, we know that $\text{pre}_{o'}[O_v] = \text{frozen} \neq o = s[O_v]$. Hence $\text{Fire}(o')$ is not applicable in s . Furthermore, $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{Fire}(o')$ because only $\text{Fire}(o)$ can change O_v from o to free , which is required because all Freeze operators (which can set O_v to frozen) require $O_v = \text{free}$ as precondition.
 - ii. $u = \text{SetO}(o')$. From the definition of SetO operators, we know that $\text{pre}_{\text{SetO}(o')}[O_v] = \text{free} \neq o = s[O_v]$. Hence $\text{Fire}(o')$ is not applicable in s . Furthermore, $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{Fire}(o')$ because only $\text{Fire}(o)$ can change O_v from o to free , which is the condition for $\text{SetO}(o')$ to become applicable.
 - iii. $u = \text{Freeze}(v, x)$ for some $x \in \mathcal{D}(v)$. We have $\text{pre}_{\text{Freeze}(v, x)}[O_v] = \text{free} \neq o = s[O_v]$ and thus Case 2(b)ii applies.
- (c) $O_v \in \text{vars}(\text{pre}_u)$ and $\text{pre}_u[O_v] \neq \text{eff}_{\text{Fire}(o)}[O_v]$. As $\text{eff}_{\text{Fire}(o)}[O_v] = \text{free}$, only fire operators $u = \text{Fire}(o')$ can be disabled (and not operators of type SetO or Freeze , as these

could only have preconditions $O_v = \text{free}$). If $v \in \text{vars}(\text{eff}_{o'})$, Case 2(a) applies. If $v \notin \text{vars}(\text{eff}_{o'})$, $v \in \text{vars}(\text{prv}_{o'})$ and Case 2(b)i applies.

- (d) $O_w \in \text{vars}(\text{pre}_u)$ for some $w \in \text{vars}(\text{prv}_{\text{Fire}(o)})$ and $\text{pre}_u[O_w] \neq \text{eff}_{\text{Fire}(o)}[O_w]$. As $\text{eff}_{\text{Fire}(o)}[O_w] = \text{free}$, only fire operators $u = \text{Fire}(o')$ can be disabled (and not operators of type SetO or Freeze , as these could only have preconditions $O_w = \text{free}$). If $w \in \text{vars}(\text{eff}_{o'})$, Case 2(a) applies. If $w \notin \text{vars}(\text{eff}_{o'})$, $w \in \text{vars}(\text{prv}_{o'})$ and Case 2(b)i applies.
 - (e) $C_w \in \text{vars}(\text{pre}_u)$ for some $w \in \text{vars}(\text{prv}_{\text{Fire}(o)})$ and $\text{pre}_u[C_w] \neq \text{eff}_{\text{Fire}(o)}[C_w]$. Only a fire operator $u = \text{Fire}(o')$ can be disabled by $C_w = \text{free}$ (and no SetC operator, because they have free as precondition). Let $\text{pre}_{\text{Fire}(o')}[C_w] := v'$. If $v' = v$, then we have $v \in \text{vars}(\text{pre}_{\text{Fire}(o')})$ and thus Case 2(a) applies. If $v' \neq v$, then $\text{pre}_{\text{Fire}(o')}[C_w] = v' \neq v = s[C_w]$ and hence $\text{Fire}(o')$ is not applicable. Furthermore, $\{\text{Fire}(o)\}$ is a necessary enabling set for $\text{Fire}(o')$, because any (fire) operator that can change C_w from v to free (which is required for SetC operators to set C_w to v') must also affect v and thus O_v . As $s[O_v] = o$, $\text{Fire}(o)$ must necessarily be applied before $\text{Fire}(o')$.
3. If $\text{Fire}(o)$ and u have conflicting effects, we must distinguish the following cases.

- (a) $v \in \text{vars}(\text{eff}_u)$ and $\text{eff}_u[v] \neq \text{eff}_{\text{Fire}(o)}[v]$. This case mirrors Case 1(a).
- (b) $O_v \in \text{vars}(\text{eff}_u)$ and $\text{eff}_u[O_v] \neq \text{eff}_{\text{Fire}(o)}[O_v]$. This case mirrors Case 1(b), with the exclusion of sub-case i., because $\text{Fire}(o)$ and $\text{Fire}(o')$ cannot disable each other via (intention) variables, as both set such variables to free .
- (c) $O_w \in \text{vars}(\text{eff}_u)$ for some variable $w \in \text{vars}(\text{prv}_{\text{Fire}(o)})$ and $\text{eff}_u[O_w] \neq \text{eff}_{\text{Fire}(o)}[O_w]$. Same case as the previous one, i.e. Case 3(b).
- (d) $C_w \in \text{vars}(\text{eff}_u)$ for some variable $w \in \text{vars}(\text{prv}_{\text{Fire}(o)})$ and $\text{eff}_u[O_w] \neq \text{eff}_{\text{Fire}(o)}[O_w]$. This case mirrors Case 1(e), with the exclusion of sub-case i., because $\text{Fire}(o)$ and $\text{Fire}(o')$ cannot disable each other via (intention) variables, as both set such variables to free .

□

Proposition 2. *Second case of Theorem 1. Let s be a state, $X \in P_s$ be an applicable operator partition of type $\text{SetO}_{v=x}$ for a variable $v \in \mathcal{V}$. Then the set $T_s := X \cup \{o' \mid o' \text{ interferes with } o'' \in X\}$ is a strong semistubborn set with the same applicable operators as X .*

Proof. We first note that $\text{SetO}_{v=x}$ contains operators $\text{SetO}(o)$ for operators $o \in \mathcal{O}$ with $\text{pre}_o[v] = x$ and one operator $\text{Freeze}(v, x)$. By definition, we have:

$$\begin{aligned}
\text{pre}_{\text{SetO}(o)/\text{Freeze}(v, x)}[O_v] &= \text{free} \\
\text{prv}_{\text{SetO}(o)/\text{Freeze}(v, x)}[v] &= x \\
\text{eff}_{\text{SetO}(o)}[O_v] &= o \\
\text{eff}_{\text{Freeze}(v, x)}[O_v] &= \text{frozen}
\end{aligned}$$

Second, note that we know the following about state s , considering that $\text{SetO}(o)$ and $\text{Freeze}(v, x)$ are applicable in s :

$$\begin{aligned}
s[v] &= x \\
s[O_v] &= \text{free}
\end{aligned}$$

We show that all operators o' that interfere with the operators from the partition X are not applicable in s . Thus the operators from X are the only applicable operators in T_s . Second, we show that for all these operators $o' \in T_s$, T_s already contains a necessary enabling set for o' in s .

Let $u \neq \text{SetO}(o)$, $\text{Freeze}(v, x)$ be an arbitrary operator interfering with $\text{SetO}(o)$ or $\text{Freeze}(v, x)$.

1. If u disables $\text{SetO}(o)$ and $\text{Freeze}(v, x)$, we only have one case: $O_v \in \text{vars}(\text{eff}_u)$ and $\text{eff}_u[O_v] \neq \text{pre}_{\text{SetO}(o)/\text{Freeze}(v, x)}[O_v]$. Note that the case $v \in \text{vars}(\text{eff}_u)$ and $\text{eff}_u[v] \neq \text{prv}_{\text{SetO}(o)/\text{Freeze}(v, x)}[v]$ can be reduced to the same case, because only fire operators can affect variables $v \in \mathcal{V}$ and if they affect variable v , they also affect O_v .

We must distinguish three possible types for u .

- (a) $u = \text{Fire}(o')$. We have $\text{pre}_{\text{Fire}(o')}[O_v] = o' \neq \text{free} = s[O_v]$ and hence $\text{Fire}(o')$ is not applicable in s . Furthermore, $\text{SetO}(o')$ must be applied before $\text{Fire}(o')$ can be applied. We claim that X is a necessary enabling set for $\text{Fire}(o')$: either $\text{SetO}(o') \in X$ in which case we are done or $\text{prv}_{\text{SetO}(o')}[v] = x'$ for $x' \neq x$ and thus any operator $\text{SetO}(o)$ from X must be applied in order to allow to change the value of v through the intended operator o . Note that it is possible that $\text{Freeze}(v, x)$ is required to allow the application of an operator which in turn fulfills a prevail-condition of $\text{Fire}(o')$. It is therefore not enough to choose the subset $\text{SetO}(o) \subseteq X$ as a necessary enabling set for $\text{Fire}(o')$ in s , but the set X is.
- (b) $u = \text{SetO}(o')$. We know $\text{prv}_{\text{SetO}(o')}[v] = x'$. If $x' = x$, $\text{SetO}(o')$ is applicable in s and $\text{SetO}(o') \in X$. If $x' \neq x$, $\text{prv}_{\text{SetO}(o')}[v] = x' \neq x = s[v]$ and hence $\text{SetO}(o')$ is not applicable in s . In the latter case, X is a necessary enabling set for $\text{SetO}(o')$ in s for the same arguments as shown in the previous Case 1(a).
- (c) $u = \text{Freeze}(v, x')$ (By assumption $u \neq \text{Freeze}(v, x)$ and hence $x' \neq x$). We have $\text{prv}_{\text{Freeze}(v, x')}[v] = x' \neq x = s[v]$ and thus $\text{Freeze}(v, x')$ is not applicable in s . X is a necessary enabling set for $\text{Freeze}(v, x')$ in s for the same reasons shown in Case 1(a).

2. If $\text{SetO}(o)$ disables u , we only have one case: $O_v \in \text{vars}(\text{pre}_u)$ and $\text{pre}_u[O_v] \neq \text{eff}_{\text{SetO}(o)}[O_v]$. We must distinguish three possible types for u .

- (a) $u = \text{Fire}(o')$. If $v \in \text{vars}(\text{eff}_{o'})$, Case 1(a) applies. If $v \notin \text{vars}(\text{eff}_{o'})$, $v \in \text{vars}(\text{prv}_{o'})$ and we have $\text{pre}_{\text{Fire}(o')}[O_v] = \text{frozen} \neq \text{free} = s[O_v]$ and thus $\text{Fire}(o')$ is not applicable. We claim that X is a necessary enabling set for $\text{Fire}(o')$ in s . We observe that $\text{Freeze}(v, x')$ for $x' = \text{prv}_{o'}[v]$ must be applied before $\text{Fire}(o')$ can be applied. If $x' = x$, $\text{Freeze}(v, x') \in X$ and we are done. If $x' \neq x$, we must first apply SetO operators in X to enable other operators to change v to x' eventually, which is prevail-condition for applying $\text{Freeze}(v, x')$, in turn requirement for application of $\text{Fire}(o')$.

- (b) $u = \text{SetO}(o')$. We have $\text{prv}_{\text{SetO}(o')}[v] = x'$ and thus Case 1(b) applies.

- (c) $u = \text{Freeze}(v, x')$. We have $\text{prv}_{\text{Freeze}(v, x')}[v] = x'$ and Case 1(c) applies.

3. If $\text{Freeze}(v, x)$ disables u , we only have one case: $O_v \in \text{vars}(\text{pre}_u)$ and $\text{pre}_u[O_v] \neq \text{eff}_{\text{SetO}(o)}[O_v]$. We must distinguish three possible types for u .

- (a) $u = \text{Fire}(o')$. If $v \in \text{vars}(\text{eff}_{o'})$, Case 1(a) applies. If $v \notin \text{vars}(\text{eff}_{o'})$, $v \in \text{vars}(\text{prv}_{o'})$ and we have $\text{prv}_{\text{Fire}(o')}[v] = x'$. If $x' = x$, $\text{Freeze}(v, x)$ does not disable $\text{Fire}(o')$. If $x' \neq x$, we have $\text{prv}_{\text{Fire}(o')}[v] = x' \neq x = s[v]$ and hence $\text{Fire}(o')$ is not applicable in s . For $\text{Fire}(o')$ to become applicable, v must change its value from x to x' . The remaining argumentation mirrors the one of Case 2(a).

- (b) $u = \text{SetO}(o')$. We have $\text{prv}_{\text{SetO}(o')}[v] = x'$ and thus Case 1b) applies.

- (c) $u = \text{Freeze}(v, x')$. We have $\text{prv}_{\text{Freeze}(v, x')}[v] = x'$ and Case 1(c) applies.

4. If $\text{SetO}(o)$ and u have conflicting effects, we only have one case: $O_v \in \text{vars}(\text{eff}_u)$ and $\text{eff}_u[O_v] \neq \text{eff}_{\text{SetO}(o)}[O_v]$. We must distinguish three possible types for u .

- (a) $u = \text{Fire}(o')$. We have $\text{pre}_{\text{Fire}(o')}[O_v] = o'$. This case mirrors Case 1(a).

- (b) $u = \text{SetO}(o')$. We have $\text{prv}_{\text{SetO}(o')}[v] = x'$. This case mirrors Case 1(b).

- (c) $u = \text{Freeze}(v, x')$. We have $\text{pre}_{\text{Freeze}(v, x')}[v] = x'$. This case mirrors Case 1(c).

5. If $\text{Freeze}(v, x)$ and u have conflicting effects, we only have one case: $O_v \in \text{vars}(\text{eff}_u)$ and $\text{eff}_u[O_v] \neq \text{eff}_{\text{Freeze}(v, x)}[O_v]$. We must distinguish two possible types for u (there is only one Freeze operator per variable and Freeze operators for different variables cannot interfere).

- (a) $u = \text{Fire}(o')$. We have $\text{pre}_{\text{Fire}(o')}[O_v] = o'$. This case mirrors Case 1(a).

- (b) $u = \text{SetO}(o')$. We have $\text{prv}_{\text{SetO}(o')}[v] = x'$. This case mirrors Case 1b).

□

Proposition 3. *Third case of Theorem 1. Let s be a state, $X \in P_s$ be an applicable operator partition of type SetC_v for a variable $v \in \mathcal{V}$. Then the set $T_s := X \cup \{o' \mid o' \text{ interferes with } o'' \in X\}$ is a strong semistubborn set with the same applicable operators as X .*

Proof. We first note that SetC_v contains operators $\text{SetC}(v, c)$ for $c \in \text{CG}(v)$. By definition, we have:

$$\begin{aligned} \text{pre}_{\text{SetC}(v, c)}[C_v] &= \text{free} \\ \text{eff}_{\text{SetC}(v, c)}[C_v] &= c \end{aligned}$$

Second, note that we know the following about state s , considering that $\text{SetC}(v, c)$ is applicable in s :

$$s[C_c] = \text{free}$$

We show that all operators o' that interfere with the operators from the partition X are not applicable in s . Thus the operators from X are the only applicable operators in T_s . Second, we show that for all these operators $o' \in T_s$, T_s already contains a necessary enabling set for o' in s .

We first observe that for an operator $\text{SetC}(v, c) \in X$, only other SetC operators for the same variable v and Fire operators may share the variable C_v with $\text{SetC}(v, c)$. Furthermore, all operators

$SetC(v, c')$ for all $c' \in CG(v)$ are already included in X (and applicable). We thus only need to consider *Fire* operators for possible interference in the following.

Let $Fire(o')$ be an arbitrary operator interfering with $SetC(v, c)$. We observe that $Fire(o')$ cannot disable $SetC(v, c)$ because it could only set C_v to *free*. There are two remaining cases.

1. If $SetC(v, c)$ disables $Fire(o')$, we have only one case: $C_v \in vars(pre_{Fire(o')})$ and $pre_{Fire(o')}[C_v] \neq eff_{SetC(v, c)}$. We have $pre_{Fire(o')}[C_v] = c'$ for a variable $c' \in vars(prv_{o'})$. If $c' = c$, $SetC(v, c)$ does not disable $Fire(o')$. If $c' \neq c$, we have $pre_{Fire(o')}[C_v] = c' \neq free = s[C_v]$ and hence $Fire(o')$ is not applicable in s . We observe that $SetC(v, c')$ must be applied before $Fire(o')$ can be applied, and because $SetC(v, c') \in X$, X is a necessary enabling set for $Fire(o')$ in s .
2. If $SetC(v, c)$ and $Fire(o')$ have conflicting effects, we know $pre_{Fire(o')}[C_v] = c'$ for a variable $c' \in vars(prv_{o'})$. Hence Case 1 applies.

□

We have shown for all operators o' that interfere with the operators of an operator partition X applicable in s that they are not applicable in s . Furthermore, X is a necessary enabling set for all such o' in s . The criteria for $T_s = X \cup \{o \mid o \text{ interferes with } o' \in X\}$ to be a strong semistubborn set in s are thus met. The overall proof for Theorem 1 follows from Propositions 1, 2, and 3.

□