

# The Relative Pruning Power of Strong Stubborn Sets and Expansion Core: Additional Proofs

**Martin Wehrle and Malte Helmert**  
University of Basel, Switzerland  
{martin.wehrle,malte.helmert}@unibas.ch

**Yusra Alkhezraji and Robert Mattmüller**  
University of Freiburg, Germany  
{alkhezry,mattmuel}@informatik.uni-freiburg.de

## Abstract

This technical report provides additional proofs and examples for “The Relative Pruning Power of Strong Stubborn Sets and Expansion Core” (Wehrle et al. 2013).

**Proposition 1.** *There is a family of planning tasks of size  $\Theta(n)$  where  $\Theta(4^n)$  states are reachable from the initial state when pruning based on EC, but only  $\Theta(2^n)$  states are reachable when pruning based on OBEC.*

**Proof:** For  $n > 0$ , let  $\Pi_n = \langle \mathcal{V}, \mathcal{O}, s_0, s_* \rangle$  be a planning task with the following components:

- $\mathcal{V} = \{a, b_1, \dots, b_n, c_1, \dots, c_n\}$  with  $\mathcal{D}_v = \{0, 1, 2\}$  for all  $v \in \mathcal{V}$
- $\mathcal{O} = B \cup C$  with  $B = \{o, o_1, \dots, o_n, \bar{o}_1, \dots, \bar{o}_n\}$  and  $C = \{o', o'_1, \dots, o'_n, \bar{o}'_1, \dots, \bar{o}'_n\}$
- $pre(o) = \{a \mapsto 0\}$ ,  $eff(o) = \{b_1 \mapsto 1, \dots, b_n \mapsto 1\}$
- $pre(o_i) = \{b_i \mapsto 1\}$ ,  $eff(o_i) = \{b_i \mapsto 2\}$  for  $1 \leq i \leq n$
- $pre(\bar{o}_i) = \{b_i \mapsto 2\}$ ,  $eff(\bar{o}_i) = \{b_i \mapsto 1\}$  for  $1 \leq i \leq n$
- $pre(o') = \{a \mapsto 0\}$ ,  $eff(o') = \{c_1 \mapsto 1, \dots, c_n \mapsto 1\}$
- $pre(o'_i) = \{c_i \mapsto 1\}$ ,  $eff(o'_i) = \{c_i \mapsto 2\}$  for  $1 \leq i \leq n$
- $pre(\bar{o}'_i) = \{c_i \mapsto 2\}$ ,  $eff(\bar{o}'_i) = \{c_i \mapsto 1\}$  for  $1 \leq i \leq n$
- $s_0 = \{a \mapsto 0, b_1 \mapsto 0, \dots, b_n \mapsto 0, c_1 \mapsto 0, \dots, c_n \mapsto 0\}$
- $s_* = \{b_1 \mapsto 2, \dots, b_n \mapsto 2\}$

We observe that variable  $a$  is trivial (i.e., has no active operators modifying it) in all reachable states. Consequently,  $s[a] = 0$  in all reachable states  $s$ .

In a reachable state, the variables  $b_i$  either all hold the value 0 (if  $o$  has never been applied) or can take on an arbitrary combination of 1s and 2s (after applying  $o$  and then some subset of  $o_i$  operators). This yields  $2^n + 1$  possible assignments to the  $b_i$  variables. Similarly, noting the symmetry between operators modifying  $b_i$  and operators modifying  $c_i$ , there are  $2^n + 1$  possible assignments to the  $c_i$  variables. It is easy to see that  $\Pi_n$  has  $(2^n + 1)^2 = 4^n + 2^{n+1} + 1$  reachable states in total. In the following, we compare EC and OBEC on this planning task.

1. Expansion core performs no pruning, leading to a total number of reachable states equal to  $4^n + 2^{n+1} + 1$ . To see this, we first observe that all operators are active in all reachable states. (This follows because the precondition of each action and the goal are reachable from every reachable state, which is easy to verify.)

Let  $s$  be an arbitrary reachable state. Then  $a$  is a potential precondition of  $b_1, \dots, b_n$  in  $s$  because of  $o$  and of  $c_1, \dots, c_n$  because of  $o'$ . Furthermore, the same operators show that  $b_1, \dots, b_n, c_1, \dots, c_n$  are potential dependents of  $a$  in  $s$ . Therefore, the potential dependency graph in  $s$  is strongly connected: every variable is connected to  $a$  in both directions. It follows that  $dc(s)$  must contain all variables and hence all applicable operators are included in  $EC(s)$ : there is no pruning by the EC method.

2. Now consider the OBEC algorithm. Let  $s$  be an arbitrary reachable non-goal state. All goals are of the form  $b_i \mapsto 2$ , so that the first operator added to  $OBEC(s)$  must be some operator  $o_i$ . Then  $OBEC(s)$  must include the corresponding operator  $\bar{o}_i$  and  $o$  because they share an effect variable with  $o_i$  (rule OBEC4). After adding  $o$ , all other operators of the form  $o_j$  and  $\bar{o}_j$  must be added because of OBEC4. At this point, no further operator can be added to  $OBEC(s)$ , so a fixed point has been reached. We see that OBEC never considers operators that modify  $c_i$  variables, while including all operators that modify  $b_i$  variables. Hence, exactly  $2^n + 1$  states are reachable when pruning with OBEC (those where  $s[c_1] = \dots = s[c_n] = 0$ ).

**Proposition 2.** *There exist planning tasks  $\Pi = \langle \mathcal{V}, \mathcal{O}, s_0, s_* \rangle$  for which neither  $OBEC(s_0)$  nor any subset of it is a strong stubborn set, no matter how the choices of disjunctive action landmarks and necessary enabling sets are resolved.*

**Proof:** We show that the statement is already true when considering  $EC(s_0)$  instead of  $OBEC(s_0)$ , which is a stronger statement because  $OBEC(s_0) \subseteq EC(s_0)$  as shown in the paper.

Let  $\Pi = \langle \mathcal{V}, \mathcal{O}, s_0, s_* \rangle$  be a planning task with the following components:

- $\mathcal{V} = \{v, w, x\}$  with  $\mathcal{D}_v = \mathcal{D}_w = \{0, 1, 2\}$  and  $\mathcal{D}_x = \{0, 1\}$

- $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$
- $pre(o_1) = \emptyset, eff(o_1) = \{v \mapsto 1, x \mapsto 1\}$
- $pre(o_2) = \{v \mapsto 1\}, eff(o_2) = \{v \mapsto 2\}$
- $pre(o_3) = \{w \mapsto 0\}, eff(o_3) = \{w \mapsto 1\}$
- $pre(o_4) = \{v \mapsto 2, w \mapsto 1\}, eff(o_4) = \{w \mapsto 2\}$
- $s_0 = \{v \mapsto 0, w \mapsto 0, x \mapsto 0\}$
- $s_* = \{v \mapsto 2, w \mapsto 2, x \mapsto 1\}$

All operators are active in  $s_0$  because  $\langle o_1, o_2, o_3, o_4 \rangle$  is a plan in  $s_0$ . Assume that  $x$  is the initialization variable that is chosen for rule EC1, so  $x \in dc(s)$ . By rule EC4, this leads to  $v \in dc(s)$  because of  $o_1$ . At this point, no rule application could extend  $dc(s)$  further: in particular, the missing variable  $w$  will not be added. (The only critical case here is EC2 with  $v' = w$  and  $o = o_4$ ; however, the rule is not applicable because  $o_4$  is not  $v$ -applicable in  $s_0$ .) We conclude that  $dc(s_0) = \{v, x\}$  and hence  $EC(s_0) = OBEC(s_0) = \{o_1, o_2\}$ .

In contrast, *every* strong stubborn set  $T_{s_0}$  must necessarily contain  $o_3$  or  $o_4$  and hence cannot be a subset of  $EC(s_0)$ . To see this, note that  $T_{s_0}$  must contain at least one applicable operator (by the completeness of the stubborn set method), and hence it contains  $o_1$  or  $o_3$ . If  $o_3 \in T_{s_0}$ , we are done. Otherwise we have  $o_1 \in T_{s_0}$  and therefore also  $o_4 \in T_{s_0}$  because  $o_1$  disables  $o_4$ . ■

## References

Wehrle, M.; Helmert, M.; Alkharaji, Y.; and Mattmüller, R. 2013. The relative pruning power of strong stubborn sets and expansion core. ICAPS 2013.