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ON THE FOUNDATIONS OF GROUNDING IN ANSWER SET PROGRAMMING

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Abstract. We provide a comprehensive elaboration of the theoretical foundations of variable instantiation, or grounding, in Answer Set Programming (ASP). Building on the semantics of ASP's modeling language, we introduce a formal characterization of grounding algorithms in terms of (fixed point) operators. A major role is played by dedicated well-founded operators whose associated models provide semantic guidance for delineating the result of grounding along with on-the-fly simplifications. We address an expressive class of logic programs that incorporates recursive aggregates and thus amounts to the scope of existing ASP modeling languages. This is accompanied with a plain algorithmic framework detailing the grounding of recursive aggregates. The given algorithms correspond essentially to the ones used in the ASP grounder *gringo*.

1. Introduction

Answer Set Programming (ASP; [46]) allows us to address knowledge-intense search and optimization problems in a greatly declarative way due to its integrated modeling, grounding, and solving workflow [31, 41]. Problems are modeled in a rule-based logical language featuring object variables, function symbols, recursion, and aggregates, among others. Moreover, the underlying nonmonotonic semantics allows us to express defaults and reachability in an easy way. A corresponding logic program is then turned into a propositional format by systematically replacing all object variables by variable-free terms. This process is called *grounding*. Finally, the actual ASP solver takes the resulting propositional version of the original program and computes its answer sets.

Given that both grounding and solving constitute the computational cornerstones of ASP, it is surprising that the importance of grounding

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has somehow been eclipsed by that of solving. This is nicely reflected by the unbalanced number of implementations. With *lparse* [58], (the grounder in) *dlv* [17], and *gringo* [27], three grounder implementations face dozens of solver implementations, among them *smodels* [56], (the solver in) *dlv* [44], *assat* [49], *cmodels* [34], *clasp* [30], *wasp* [2] just to name the major ones. What caused this imbalance? One reason may consist in the high expressiveness of ASP's modeling language that allows modern grounders to mimic universal Turing machines (cf. [27]). Another may lie in the popular viewpoint that grounding amounts to database materialization, and thus that most fundamental research questions have been settled. And finally the semantic foundations of full-featured ASP modeling languages have been established only recently [24, 37], revealing the semantic gap to the just mentioned idealized understanding of grounding. In view of this, research on grounding focused on algorithm and system design [17, 27] and the characterization of language fragments guaranteeing finite propositional representations [9, 32, 45, 58].

As a consequence, the theoretical foundations of grounding are much less explored than those of solving. While there are several alternative ways to characterize the answer sets of a logic program [47], and thus the behavior of a solver, we still lack indepth formal characterizations of the input-output behavior of ASP grounders. Although we can describe the resulting propositional program up to semantic equivalence, we have no formal means to delineate the actual set of rules.

To this end, grounding involves some challenging intricacies. First of all, the entire set of systematically instantiated rules is infinite in the worst — yet not uncommon — case. For a simple example, consider the program:

$$p(a)$$
$$p(X) \leftarrow p(f(X))$$

This program induces an infinite set of variable-free terms, viz. a, $f(a)$, $f(f(a)), \ldots$, that leads to an infinite propositional program by systematically replacing variable X by all these terms in the second rule. On the other hand, modern grounders only produce the fact $p(a)$ and no instances of the second rule, which is semantically equivalent to the infinite program. As well, ASP's modeling language comprises (possibly recursive) aggregates, whose systematic grounding may be infinite in itself. To illustrate this, let us extend the above program with the rule

$$q \leftarrow \#count\{X : p(X)\} = 1 \tag{1}$$

deriving q when the number of satisfied instances of p is one. Analogous to above, the systematic instantiation of the aggregate's element results in an infinite set. Again, a grounder is able to infer a fact. That is, it detects that the set amounts to a singleton that satisfies the aggregate. After removing the rule's (satisfied) antecedent, it produces the fact q. In fact, a solver expects a finite set of propositional rules including aggregates over finitely

many objects only. Hence, in practice, the characterization of the grounding result boils down to identifying a finite yet semantically equivalent set of rules (whenever possible). Finally, in practice, grounding involves simplifications whose application depends on the ordering of rules in the input. In fact, shuffling a list of propositional rules only affects the order in which a solver enumerates answer sets, whereas shuffling a logic program before grounding may lead to different though semantically equivalent sets of rules. To see this, consider the program:

$$\begin{array}{lll} p(X) \leftarrow \neg q(X) \wedge u(X) & \qquad u(1) \quad u(2) & \\ q(X) \leftarrow \neg p(X) \wedge v(X) & \qquad\quad\; v(2) \quad v(3) & \end{array}$$

This program has two answer sets; both contain $p(1)$ and $q(3)$, while one contains $q(2)$ and the other $p(2)$. Systematically grounding the program yields the obvious four rules. However, depending upon the order, the rules are passed to a grounder, it already produces either the fact $p(1)$ or $q(3)$ via simplification. Clearly, all three programs are distinct but semantically equivalent in sharing the above two answer sets.

Building on the semantics of ASP modeling languages [24,37], we elaborate upon the foundations of ASP grounding and introduce a formal characterization of grounding algorithms in terms of (fixed point) operators. A major role is played by dedicated well-founded operators whose associated models provide semantic guidance for delineating the result of grounding along with on-the-fly simplifications. We address an expressive class of logic programs that incorporates recursive aggregates and thus amounts to the scope of existing ASP modeling languages [25]. This is accompanied with an algorithmic framework detailing the grounding of recursive aggregates. The given grounding algorithms correspond essentially to the ones used in the ASP grounder *gringo* [27]. In this way, our framework provides a formal characterization of one of the most widespread grounding systems.

Modern grounders like (the one in) *dlv* [17] or *gringo* [27] are based on database evaluation techniques [1, 63]. Grounding is seen as an iterative bottom-up process guided by the successive expansion of a program's Herbrand base, that is, the set of variable-free atoms constructible from the signature of the rules at hand. This process is repeated until a fixed point is reached where no further atoms can be added. During this process, a ground rule is produced if its positive body atoms belong to the current Herbrand base, in which case its head atom is added to the current Herbrand base.

From an algorithmic perspective, we show how a grounding framework (relying upon semi-naive database evaluation techniques) can be extended to incorporate recursive aggregates. An example of such an aggregate is shown in Table 1, giving an encoding of the *Company Controls Problem* [52]: A company X controls a company Y, if X directly or indirectly controls more than 50% of the shares of Y. The aggregate $\#\text{sum}^+$ implements summation over positive integers. Notably, it takes part in the recursive definition of predicate *controls* in Table 1. A corresponding problem instance is given in

$$\begin{aligned} \mathit{controls}(X,Y) \leftarrow \#\mathrm{sum}^+\{S : \mathit{owns}(X,Y,S);& \\ S,Z : \mathit{controls}(X,Z) \wedge \mathit{owns}(Z,Y,S)\} > 50& \\ \wedge\ \mathit{company}(X) \wedge \mathit{company}(Y) \wedge X \neq Y& \end{aligned}$$

TABLE 1. Company Controls Encoding

Table 2. Note that a systematic instantiation of the four variables in Table 1

$\mathit{company}(c_1)$	$\mathit{company}(c_2)$	$\mathit{company}(c_3)$	$\mathit{company}(c_4)$
$\mathit{owns}(c_1,c_2,60)$	$\mathit{owns}(c_1,c_3,20)$	$\mathit{owns}(c_2,c_3,35)$	$\mathit{owns}(c_3,c_4,51)$

TABLE 2. Company Controls Instance

with the eight constants in Table 2 results in 64 ground rules. However, taken together, the encoding and the instance are equivalent to the program in Table 3, which consists of four ground rules only. In fact, all literals in Table 3

$$\begin{aligned} \mathit{controls}(c_1,c_2) \leftarrow \#\mathrm{sum}^+\{60 : \mathit{owns}(c_1,c_2,60)\} > 50& \\ \wedge\ \mathit{company}(c_1) \wedge \mathit{company}(c_2) \wedge c_1 \neq c_2& \\ \mathit{controls}(c_3,c_4) \leftarrow \#\mathrm{sum}^+\{51 : \mathit{owns}(c_3,c_4,51)\} > 50& \\ \wedge\ \mathit{company}(c_3) \wedge \mathit{company}(c_4) \wedge c_3 \neq c_4& \\ \mathit{controls}(c_1,c_3) \leftarrow \#\mathrm{sum}^+\{20 : \mathit{owns}(c_1,c_3,20);& \\ 35, c_2 : \mathit{controls}(c_1,c_2) \wedge \mathit{owns}(c_2,c_3,35)\} > 50& \\ \wedge\ \mathit{company}(c_1) \wedge \mathit{company}(c_3) \wedge c_1 \neq c_3& \\ \mathit{controls}(c_1,c_4) \leftarrow \#\mathrm{sum}^+\{51, c_3 : \mathit{controls}(c_1,c_3) \wedge \mathit{owns}(c_3,c_4,51)\} > 50& \\ \wedge\ \mathit{company}(c_1) \wedge \mathit{company}(c_4) \wedge c_1 \neq c_4& \end{aligned}$$

TABLE 3. Relevant Grounding of Company Controls

can be completely evaluated in view of the problem instance, which moreover allows us to evaluate the aggregate atoms, so that the grounding of the above company controls instance boils down to the four facts $\mathit{controls}(c_1,c_2)$, $\mathit{controls}(c_3,c_4)$, $\mathit{controls}(c_1,c_3)$, and $\mathit{controls}(c_1,c_4)$.

These four facts are also obtained as output from the grounder *gringo*.

Our paper is organized as follows.

Section 2 lays the basic foundations of our approach. We start in Section 2.1 by recalling definitions of (monotonic) operators on lattices; they constitute the basic building blocks of our characterization of grounding algorithms. We then review infinitary formulas along with their stable and

well-founded semantics in Sections 2.2, 2.3 and 2.4, respectively. In this context, we explore several operators and define a class of infinitary logic programs that allows us to capture full-featured ASP languages with (recursive) aggregates. Interestingly, we have to resort to concepts borrowed from ID-logic [8, 60] to obtain monotonic operators that are indispensable for capturing iterative algorithms. Finally, we define in Section 2.5 our concept of program simplification and elaborate upon its semantic properties.

Section 3 is dedicated to the formal foundations of component-wise grounding. As mentioned, each rule is instantiated in the context of the current Herbrand base. In addition, grounding has to take subsequent atom definitions into account. To this end, we extend well-known operators and resulting semantic concepts with contextual information, usually captured by two- and four-valued interpretations, respectively, and elaborate upon their formal properties that are relevant to grounding. In turn, we generalize the contextual operators and semantic concepts to sequences of programs in order to reflect component-wise grounding. The major emerging concept is essentially a well-founded model for program sequences that takes backward and forward contextual information into account. This model-theoretic concept can be used for governing an ideal grounding process.

Section 4 turns to logic programs with variables and aggregates. We align the semantics of such aggregate programs with that of Ferraris [19] but consider infinitary formulas [37]. In view of grounding aggregates, however, we introduce an alternative translation of aggregates that is strongly equivalent to that of Ferraris but permits stronger propagation. As a result, we obtain a translation of finite, non-ground logic programs with aggregates into the class of infinitary, ground logic programs defined in Section 2. Such infinitary programs can be turned into finitary ones by means of the program simplification introduced in Section 2.5 under certain conditions.

Section 5 further refines our semantic approach to reflect actual grounding processes. To this end, we define the concept of an instantiation sequence based on rule dependencies. We then use the contextual operators of Section 3 to define approximate models of instantiation sequences. While approximate models are in general less precise than well-founded ones, they are better suited for on-the-fly grounding along an instantiation sequence. Nonetheless they are strong enough to allow for completely evaluating stratified programs.

Section 6 lays out the basic algorithms for grounding rules, components, and entire programs and characterizes their output in terms of the semantic concepts developed in the previous sections. Of particular interest is the treatment of aggregates, which are decomposed into dedicated normal rules before grounding, and re-assembled afterwards. This allows us to ground rules with aggregates by means of grounding algorithms for normal rules.

The previous sections focus on the theoretical and algorithmic cornerstones of grounding. Section 7 refines these concepts by further detailing aggregate propagation, algorithm specifics, and the treatment of language constructs from *gringo*'s input language.

Finally, we relate our contributions to the state of the art in Section 8 and summarize it in Section 9.

The developed approach is implemented in *gringo* series 4 and 5. However, to ease comprehensibility, we have moreover implemented the presented approach in *μ-gringo*[1] and equipped it with means for retracing the developed concepts during grounding. This system may also enable some readers to construct and to experiment with own grounder extensions.

This paper draws on material presented during an invited talk at the third workshop on grounding, transforming, and modularizing theories with variables [29].

2. Foundations

2.1. Operators on lattices

This section recalls basic concepts on operators on complete lattices.

A *complete lattice* is a partially ordered set $(L, \leq)$ in which every subset $S \subseteq L$ has a greatest lower bound and a least upper bound in $(L, \leq)$.

An *operator* O on lattice $(L, \leq)$ is a function from L to L. It is *monotone* if $x \leq y$ implies $O(x) \leq O(y)$ for each $x, y \in L$; and it is *antimonotone* if $x \leq y$ implies $O(y) \leq O(x)$ for each $x, y \in L$.

Let O be an operator on lattice $(L, \leq)$. A *prefixed point* of O is an $x \in L$ such that $O(x) \leq x$. A *postfixed point* of O is an $x \in L$ such that $x \leq O(x)$. A *fixed point of* O is an $x \in L$ such that $x = O(x)$, i.e., it is both a prefixed and a postfixed point.

Theorem 1 (Knaster-Tarski; [59])**.** *Let O be a monotone operator on complete lattice $(L, \leq)$. Then, we have the following properties:*

(a) *Operator O has a least fixed and prefixed point which are identical.*
(b) *Operator O has a greatest fixed and postfixed point which are identical.*
(c) *The fixed points of O form a complete lattice.*

2.2. Formulas and Interpretations

We begin with a propositional signature Σ consisting of a set of atoms. Following [60], we define the sets $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of formulas as follows:

- $\mathcal{F}_0$ is the set of all propositional atoms in Σ,
- $\mathcal{F}_{i+1}$ is the set of all elements of $\mathcal{F}_i$, all expressions $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ with $\mathcal{H} \subseteq \mathcal{F}_i$, and all expressions $F \to G$ with $F, G \in \mathcal{F}_i$.

The set $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$ contains all *(infinitary propositional) formulas* over Σ.

In the following, we use the following shortcuts:

- $\top = \emptyset^\wedge$ and $\bot = \emptyset^\vee$,
- $\neg F = F \to \bot$ where F is a formula, and
- $F \wedge G = \{F, G\}^\wedge$ and $F \vee G = \{F, G\}^\vee$ where F and G are formulas.

[1]The *μ-gringo* system is available at `https://github.com/potassco/mu-gringo`.

An occurrence of a subformula in a formula is called *positive*, if the number of implications containing that occurrence in the antecedent is even, and *strictly positive* if that number is zero; if that number is odd the occurrence is *negative*. The sets F^+ and F^- gather all atoms occurring positively or negatively in formula F, respectively; if applied to a set of formulas, both expressions stand for the union of the respective atoms in the formulas. Also, we define $F^\pm = F^+ \cup F^-$ as the set of all atoms occurring in F.

A *two-valued interpretation* over signature Σ is a set I of propositional atoms such that $I \subseteq \Sigma$. Atoms in an interpretation I are considered *true* and atoms in $\Sigma \setminus I$ as *false*. The set of all interpretations together with the $\subseteq$ relation forms a complete lattice.

The satisfaction relation between interpretations and formulas is defined as follows:

- $I \models a$ for atoms a if $a \in I$,
- $I \models \mathcal{H}^\wedge$ if $I \models F$ for all $F \in \mathcal{H}$,
- $I \models \mathcal{H}^\vee$ if $I \models F$ for some $F \in \mathcal{H}$, and
- $I \models F \to G$ if $I \not\models F$ or $I \models G$.

An interpretation I is a *model* of a set $\mathcal{H}$ of formulas, written $I \models \mathcal{H}$, if it satisfies each formula in the set.

In the following, all atoms, formulas, and interpretations operate on the same (implicit) signature, unless mentioned otherwise.

2.3. **Stable models**

Our terminology in this section keeps following the one in [60].

The *reduct* F^I of a formula F w.r.t. an interpretation I is defined as:

- $\bot$ if $I \not\models F$,
- a if $I \models F$ and $F = a \in \mathcal{F}_0$,
- $\{G^I \mid G \in \mathcal{H}\}^\wedge$ if $I \models F$ and $F = \mathcal{H}^\wedge$,
- $\{G^I \mid G \in \mathcal{H}\}^\vee$ if $I \models F$ and $F = \mathcal{H}^\vee$, and
- $G^I \to H^I$ if $I \models F$ and $F = G \to H$.

An interpretation I is a *stable model* of a formula F if it is among the (set inclusion) minimal models of F^I.

Note that the reduct removes (among other unsatisfied subformulas) all occurrences of atoms that are false in I. Thus, the satisfiability of the reduct does not depend on such atoms, and all minimal models of F^I are subsets of I. Hence, if I is a stable model of F, then it is the only minimal model of F^I.

Sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of infinitary formulas are *equivalent* if they have the same stable models and *classically equivalent* if they have the same models; they are *strongly equivalent* if, for any set $\mathcal{H}$ of infinitary formulas, $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_1 \cup \mathcal{H}$ are equivalent. As shown in [36], this also allows for replacing a part of any formula with a strongly equivalent formula without changing the

set of stable models. We use the following reduct-based characterization of strong equivalence [62].

Proposition 2. *Sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of infinitary formulas are strongly equivalent iff $\mathcal{H}_1^I$ and $\mathcal{H}_2^I$ are classically equivalent for all two-valued interpretations I.*

See proof on page 65.

In the following, we do not consider stable models of arbitrary formulas but formulas with atoms as head and formulas as body. Accordingly, an *$\mathcal{F}$-program* is set of *rules* of form $h \leftarrow F$ where $h \in \mathcal{F}_0$ and $F \in \mathcal{F}$. We use $H(h \leftarrow F) = h$ to refer to rule *heads* and $B(h \leftarrow F) = F$ to refer to rule *bodies*. We extend this to programs by letting $H(P) = \{H(r) \mid r \in P\}$ and $B(P) = \{B(r) \mid r \in P\}$.

An interpretation I is a model of P, written $I \models P$, if $I \models B(r) \rightarrow H(r)$ for all $r \in P$. The latter is also written as $I \models r$. We define the reduct of an $\mathcal{F}$-program P w.r.t. an interpretation I as $P^I = \{r^I \mid r \in P\}$ where $r^I = H(r) \leftarrow B(r)^I$. Note that r^I leaves the head of r intact and only reduces its body. As above, an interpretation I is a stable model of P if I is among the minimal models of P^I.

This program-oriented reduct yields the same stable models as obtained by applying the full reduct to the corresponding infinitary formula.

Proposition 3. *The $\mathcal{F}$-program P has the same stable models as the formula $\{B(r) \rightarrow H(r) \mid r \in P\}^\wedge$.*

See proof on page 65.

For programs, Truszczyński introduces in [60] an alternative reduct, replacing each negatively occurring atom with $\bot$, if it is falsified, and with $\top$, otherwise. More precisely, the so-called ID-*reduct* F_I of a formula F w.r.t. an interpretation I is defined as

$$
\begin{array}{rlrl}
a_I &= a & a_{\bar{I}} &= \top \text{ if } a \in I \\
 & & a_{\bar{I}} &= \bot \text{ if } a \notin I \\
\mathcal{H}_I^\wedge &= \{F_I \mid F \in \mathcal{H}\}^\wedge & \mathcal{H}_{\bar{I}}^\wedge &= \{F_{\bar{I}} \mid F \in \mathcal{H}\}^\wedge \\
\mathcal{H}_I^\vee &= \{F_I \mid F \in \mathcal{H}\}^\vee & \mathcal{H}_{\bar{I}}^\vee &= \{F_{\bar{I}} \mid F \in \mathcal{H}\}^\vee \\
(F \rightarrow G)_I &= F_{\bar{I}} \rightarrow G_I & (F \rightarrow G)_{\bar{I}} &= F_I \rightarrow G_{\bar{I}}
\end{array}
$$

where a is an atom, $\mathcal{H}$ a set of formulas, and F and G are formulas.

The ID-reduct of an $\mathcal{F}$-program P w.r.t. an interpretation I is $P_I = \{r_I \mid r \in P\}$ where $r_I = H(r) \leftarrow B(r)_I$. As with r^I, the transformation of r into r_I leaves the head of r unaffected.

Example 1. Consider the program containing the single rule

$$p \leftarrow \neg\neg p.$$

We get the following reduced programs w.r.t. interpretations $\emptyset$ and $\{p\}$:

$$\{p \leftarrow \neg\neg p\}^{\emptyset} = \{p \leftarrow \bot\} \qquad \{p \leftarrow \neg\neg p\}^{\{p\}} = \{p \leftarrow \neg\bot\}$$
$$\{p \leftarrow \neg\neg p\}_{\emptyset} = \{p \leftarrow \neg\neg p\} \quad = \quad \{p \leftarrow \neg\neg p\}_{\{p\}} = \{p \leftarrow \neg\neg p\}$$

Note that both reducts leave the rule's head intact.

Extending the definition of positive occurrences, we define a formula as *(strictly) positive* if all its atoms occur (strictly) positively in the formula. We define an $\mathcal{F}$-program as (strictly) positive if all its rule bodies are (strictly) positive.

Proposition 4. *Let F be a formula, and I and J be interpretations. If F is positive and $I \subseteq J$, then $I \models F$ implies $J \models F$.*

See proof on page 65.

The following two propositions shed some light on the two types of reducts.

Proposition 5. *Let F be a formula, and I and J be interpretations. Then,*

(a) *if F is positive then F^I is positive,*
(b) *$I \models F$ iff $I \models F^I$,*
(c) *if F is strictly positive and $I \subseteq J$ then $I \models F$ iff $I \models F^J$.*

See proof on page 65.

Proposition 6. *Let F be a formula, and I, J, and X be interpretations. Then,*

(a) *F_I is positive,*
(b) *$I \models F$ iff $I \models F_I$,*
(c) *if F is positive then $F = F_I$, and*
(d) *if $I \subseteq J$ then $X \models F_J$ implies $X \models F_I$.*

See proof on page 66.

As put forward in [64], we may associate with each program P its *one-step provability operator* T_P, defined for any interpretation X as

$$T_P(X) = \{H(r) \mid r \in P, X \models B(r)\}.$$

Proposition 7 ([60])**.** *Let P be a positive $\mathcal{F}$-program. Then, the operator T_P is monotone.*

Fixed points of T_P are models of P guaranteeing that each contained atom is supported by some rule in P; prefixed points of T_P correspond to the models of P. According to Theorem 1 (a), the T_P operator has a least fixed point for positive $\mathcal{F}$-programs. We refer to this fixed point as the *least model* of P, and write it as $LM(P)$.

Now, in view of Proposition 6 (a), any ID-reduct P_I of a program w.r.t. an interpretation I possesses a least model $LM(P_I)$. This gives rise to the

following definition of a stable operator [60]: Given an $\mathcal{F}$-program P, its ID-*stable operator* is defined for any interpretation I as

$$S_P(I) = LM(P_I).$$

The fixed points of S_P are the ID-*stable models* of P.

Note that neither the program reduct P^I nor the formula reduct F^I guarantee (least) models. Also, stable models and ID-stable models do not coincide in general.

Example 2. Reconsider the program from Example 1, comprising rule

$$p \leftarrow \neg\neg p.$$

This program has the two stable models $\emptyset$ and $\{p\}$, but the empty model is the only ID-stable model.

Proposition 8 ([60])**.** *Let P be an $\mathcal{F}$-program.*
Then, the ID-*stable operator S_P is antimonotone.*

No analogous antimonotone operator is obtainable for $\mathcal{F}$-programs by using the program reduct P^I (and for general theories with the formula reduct F^I). To see this, reconsider Example 2 along with its two stable models $\emptyset$ and $\{p\}$. Given that both had to be fixed points of such an operator, it would behave monotonically on $\emptyset$ and $\{p\}$.

Truszczyński identifies in [60] a class of programs for which stable models and ID-stable models coincide. The set $\mathcal{N}$ consists of all formulas F such that any implication in F has $\bot$ as consequent and no occurrences of implication in its antecedent. An $\mathcal{N}$-program consist of rules of form $h \leftarrow F$ where $h \in \mathcal{F}_0$ and $F \in \mathcal{N}$.

Proposition 9 ([60])**.** *Let P be an $\mathcal{N}$-program.*
Then, the stable and ID-*stable models of P coincide.*

Note that a positive $\mathcal{N}$-program is also strictly positive.

2.4. **Well-founded models**

In the following, we deal with pairs of sets and extend the basic set relations and operations accordingly. Given sets I', I, J', J, and X, we define:

- $(I', J') \bar{\prec} (I, J)$ if $I' \prec I$ and $J' \prec J$ for $(\bar{\prec}, \prec) \in \{(\sqsubset, \subset), (\sqsubseteq, \subseteq)\}$
- $(I', J') \bar{\circ} (I, J) = (I' \circ I, J' \circ J)$ for $(\bar{\circ}, \circ) \in \{(\sqcup, \cup), (\sqcap, \cap), (\diagdown, \setminus)\}$
- $(I, J) \bar{\circ} X = (I, J) \bar{\circ} (X, X)$ for $\bar{\circ} \in \{\sqcup, \sqcap, \diagdown\}$

Our terminology in this section follows the one in [61].

A *four-valued interpretation* over signature Σ is represented by a pair $(I, J) \sqsubseteq \Sigma \times \Sigma$ where I stands for *certain* and J for *possible* atoms. Intuitively, an atom that is

- certain and possible is *true*,
- certain but not possible is *inconsistent*,
- not certain but possible is *unknown*, and

- not certain and not possible is *false*.

A four-valued interpretation (I', J') is more precise than a four-valued interpretation (I, J), written $(I, J) \leq_p (I', J')$, if $I \subseteq I'$ and $J' \subseteq J$. The precision ordering also has an intuitive reading: the more atoms are certain or the fewer atoms are possible, the more precise is an interpretation. The least precise four-valued interpretation over Σ is $(\emptyset, \Sigma)$. As with two-valued interpretations, the set of all four-valued interpretations over a signature Σ together with the relation $\leq_p$ forms a complete lattice. A four-valued interpretation is called *inconsistent* if it contains an inconsistent atom; otherwise, it is called *consistent*. It is *total* whenever it makes all atoms either true or false.

In analogy to [61], we define the ID-*well-founded operator* of an $\mathcal{F}$-program P for any four-valued interpretation (I, J) as

$$W_P(I, J) = (S_P(J), S_P(I)).$$

This operator is monotone w.r.t. the precision ordering $\leq_p$. Hence, by Theorem 1 (a), W_P has a least fixed point, which defines the ID-*well-founded model* of P, also written as $WM(P)$. In what follows, we drop the prefix 'ID' and simply refer to the ID-well-founded model of a program as its well-founded model. (We keep the distinction between stable and ID-stable models.)

Any well-founded model (I, J) of an $\mathcal{F}$-program P satisfies $I \subseteq J$.

Lemma 10. *Let P be an $\mathcal{F}$-Program.*
Then, the well-founded model $WM(P)$ of P is consistent.

See proof on page 67.

Example 3. Consider program P_3 consisting of the following rules:

$$a$$
$$b \leftarrow a$$
$$c \leftarrow \neg b$$
$$d \leftarrow c$$
$$e \leftarrow \neg d$$

We compute the well-founded model of P_3 starting from $(\emptyset, \Sigma)$:

$$S_P(\Sigma) = \{a, b\} \qquad S_P(\emptyset) = \{a, b, c, d, e\}$$
$$S_P(\{a, b, c, d, e\}) = \{a, b\} \qquad S_P(\{a, b\}) = \{a, b, e\}$$
$$S_P(\{a, b, e\}) = \{a, b, e\} \qquad S_P(\{a, b\}) = \{a, b, e\}$$
$$S_P(\{a, b, e\}) = \{a, b, e\} \qquad S_P(\{a, b, e\}) = \{a, b, e\}$$

The well-founded model of P_3 is $(\{a, b, e\}, \{a, b, e\})$.

Unlike general $\mathcal{F}$-programs, the class of $\mathcal{N}$-programs warrants the same stable and ID-stable models for each of its programs. Unfortunately, $\mathcal{N}$-programs are too restricted for our purpose (for instance, for capturing

aggregates in rule bodies[2]). To this end, we define a more general class of programs and refer to them as $\mathcal{R}$-programs. Although ID-stable models of $\mathcal{R}$-programs may differ from their stable models (see below), their well-founded models encompass both stable and ID-stable models. Thus, well-founded models can be used for characterizing stable model-preserving program transformations. In fact, we see in Section 2.5 that the restriction of $\mathcal{F}$- to $\mathcal{R}$-programs allows us to provide tighter semantic characterizations of program simplifications.

We define $\mathcal{R}$ to be the set of all formulas F such that implications in F have no further occurrences of implications in their antecedents. Then, an *$\mathcal{R}$-program* consist of rules of form $h \leftarrow F$ where $h \in \mathcal{F}_0$ and $F \in \mathcal{R}$. As with $\mathcal{N}$-programs, a positive $\mathcal{R}$-program is also strictly positive.

Our next result shows that ID-well-founded models can be used for approximating regular stable models of $\mathcal{R}$-programs.

Theorem 11. *Let P be an $\mathcal{R}$-program and (I, J) be the well-founded model of P.*

If X is a stable model of P, then $I \subseteq X \subseteq J$.

See proof on page 69.

Example 4. Consider the $\mathcal{R}$-program P_4:[3]

$$c \leftarrow (b \rightarrow a) \qquad a \leftarrow b$$
$$a \leftarrow c \qquad b \leftarrow a$$

Observe that $\{a, b, c\}$ is the only stable model of P_4, the program does not have any ID-stable models, and the well-founded model of P_4 is $(\emptyset, \{a, b, c\})$. In accordance with Theorem 11, the stable model of P_4 is enclosed in the well-founded model.

Note that the ID-reduct handles $b \rightarrow a$ the same way as $\neg b \vee a$. In fact, the program obtained by replacing

$$c \leftarrow (b \rightarrow a)$$

with

$$c \leftarrow \neg b \vee a$$

is an $\mathcal{N}$-program and has neither stable nor ID-stable models.

Further, note that the program in Example 2 is not an $\mathcal{R}$-program, whereas the one in Example 3 is an $\mathcal{R}$-program.

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[2]Ferraris' semantics [19] of aggregates introduces implications, which results in rules beyond the class of $\mathcal{N}$-programs.

[3]The choice of the body $b \rightarrow a$ is not arbitrary since it can be seen as representing the aggregate $\#\text{sum}\{1 : a, -1 : b\} \geq 0$.

2.5. **Program simplification**

In this section, we define a concept of program simplification and show how its result can be characterized by the semantic means from above. In particular, we delineate its preservation of well-founded and stable models.

Definition 1. Let P be an $\mathcal{F}$-program, and (I, J) be a four-valued interpretation.

We define the simplification of P w.r.t. (I, J) as

$$P^{(I,J)} = \{r \in P \mid J \models B(r)_I\}.$$

For simplicity, we drop parentheses and we write $P^{I,J}$ instead of $P^{(I,J)}$ whenever clear from context.

The program simplification $P^{I,J}$ acts as a filter eliminating inapplicable rules that fail to satisfy the condition $J \models B(r)_I$. That is, first, all negatively occurring atoms in $B(r)$ are evaluated w.r.t. the certain atoms in I and replaced accordingly by $\bot$ and $\top$, respectively. Then, it is checked whether the reduced body $B(r)_I$ is satisfiable by the possible atoms in J. Only in this case, the rule is kept in $P^{I,J}$. No simplifications are applied to the remaining rules. This is illustrated in Example 5 below.

Note that for an $\mathcal{F}$-program P the head atoms in $P^{I,J}$ correspond to the result of applying the provability operator of program P_I to the possible atoms in J:

Lemma 12. *Let P be an $\mathcal{F}$-program and (I, J) be a four-valued interpretation.*

Then, we have $H(P^{I,J}) = T_{P_I}(J)$.

See proof on page 70.

Our next result shows that programs simplified with their well-founded model maintain this model; it is based on the following proposition.

Proposition 13. *Let P be an $\mathcal{F}$-program and (I, J) be the well-founded model of P.*

Then, we have

(a) *$S_{P^{I,J}}(I') = J$ for all $I' \subseteq I$, and*
(b) *$S_{P^{I,J}}(J') = S_P(J')$ for all $J \subseteq J'$.*

See proof on page 70.

Theorem 14. *Let P be an $\mathcal{F}$-program and (I, J) be the well-founded model of P.*

Then, P and $P^{I,J}$ have the same well-founded model.

See proof on page 71.

Example 5. In Example 3, we computed the well-founded model $(\{a, b, e\}, \{a, b, e\})$ of P_3. With this, we obtain the simplified program $P_3' = P_3^{\{a,b,e\},\{a,b,e\}}$

$$a$$
$$b \leftarrow a$$
$$e \leftarrow \neg d$$

after dropping $c \leftarrow \neg b$ and $d \leftarrow c$.

Next, we check that the well-founded model of P_3' corresponds to the well-founded model of P_3:

$$S_{P_3'}(\Sigma) = \{a, b\} \qquad S_{P_3'}(\emptyset) = \{a, b, e\}$$
$$S_{P_3'}(\{a, b, e\}) = \{a, b, e\} \qquad S_{P_3'}(\{a, b\}) = \{a, b, e\}$$
$$S_{P_3'}(\{a, b, e\}) = \{a, b, e\} \qquad S_{P_3'}(\{a, b, e\}) = \{a, b, e\}$$

We observe that it takes two applications of the well-founded operator to obtain the well-founded model. This could be reduced to one step if atoms false in the well-founded model would be removed from the negative bodies by the program simplification. Keeping them is a design decision with the goal to simplify notation in the following.

The next series of results further elaborates on semantic invariants guaranteed by our concept of program simplification. The first result shows that it preserves all stable models between the sets used for simplification.

Theorem 15. *Let P be an $\mathcal{F}$-program, and I, J and X be two-valued interpretations.*

If $I \subseteq X \subseteq J$, then X is a stable model of P iff X is a stable model of $P^{I,J}$.

See proof on page 72.

As a consequence, we obtain that $\mathcal{R}$-programs simplified with their well-founded model also maintain stable models.

Corollary 16. *Let P be an $\mathcal{R}$-program and (I, J) be the well-founded model of P.*

Then, P and $P^{I,J}$ have the same stable models.

See proof on page 72.

For instance, the $\mathcal{R}$-program in Example 3 and its simplification in Example 5 have the same stable model. Unlike this, the rule $p \leftarrow \neg\neg p$ from Example 2 induces two stable models, while its simplification w.r.t. its well-founded model $(\emptyset, \emptyset)$ yields an empty program admitting the empty stable model only.

The next two results show that any program between the original and its simplification relative to its well-founded model preserves the well-founded model, and that this extends to all stable models for $\mathcal{R}$-programs.

Theorem 17. *Let P and Q be $\mathcal{F}$-programs, and (I, J) be the well-founded model of P.*

If $P^{I,J} \subseteq Q \subseteq P$, then P and Q have the same well-founded models.

See proof on page 72.

Corollary 18. *Let P and Q be $\mathcal{R}$-programs, and (I, J) be the well-founded model of P.*

If $P^{I,J} \subseteq Q \subseteq P$, then P and Q are equivalent.

See proof on page 73.

3. Splitting

One of the first steps during grounding is to group rules into components suitable for successive instantiation. This amounts to splitting a logic program into a sequence of subprograms. The rules in each such component are then instantiated with respect to the Herbrand base of all previous components, starting with some component consisting of facts only. In other words, grounding is always performed relative to a set of context atoms. Moreover, atoms found to be true or false can be used to apply on-the-fly simplifications.

Accordingly, this section parallels the above presentation by extending the respective formal concepts with contextual information provided by context atoms in a two- and four-valued setting. We then assemble the resulting concepts to enable their consecutive application to sequences of subprograms. Interestingly, the resulting notion of splitting is more general then the traditional concept [48] since it allows us to partition rules in an arbitrary way. In the following, we use the superscript $^{\mathrm{c}}$ to indicate contextual interpretations.

To begin with, we extend the one-step provability operator accordingly.

Definition 2. Let P be an $\mathcal{F}$-program and I^{c} be a two-valued interpretation.

For any two-valued interpretation I, we define the *one-step provability operator of P relative to I^{c}* as

$$T_P^{I^{\mathrm{c}}}(I) = T_P(I^{\mathrm{c}} \cup I).$$

A prefixed point of $T_P^{I^{\mathrm{c}}}$ is a also a prefixed point of T_P. Thus, each prefixed point of $T_P^{I^{\mathrm{c}}}$ is a model of P but not necessarily the other way round.

To see this, consider program $P = \{a \leftarrow b\}$. We have $T_P(\emptyset) = \emptyset$ and $T_P^{\{b\}}(\emptyset) = \{a\}$. Hence, $\emptyset$ is a (pre)fixed point of T_P but not of $T_P^{\{b\}}$ since $\{a\} \not\subseteq \emptyset$. The set $\{a\}$ is a prefixed point of both operators.

Alternatively, the incorporation of context atoms can also be seen as a form of partial evaluation applied to the underlying program.

Definition 3. Let I^{c} be a two-valued interpretation.

We define the *partial evaluation* of an $\mathcal{F}$-formula w.r.t. I^c as follows:

$$pe_{I^c}(a) = \top \text{ if } a \in I^c \qquad pe_{\bar{I}^c}(a) = a$$
$$pe_{I^c}(a) = a \text{ if } a \notin I^c$$
$$pe_{I^c}(\mathcal{H}^\wedge) = \{pe_{I^c}(F) \mid F \in \mathcal{H}\}^\wedge \qquad pe_{\bar{I}^c}(\mathcal{H}^\wedge) = \{pe_{\bar{I}^c}(F) \mid F \in \mathcal{H}\}^\wedge$$
$$pe_{I^c}(\mathcal{H}^\vee) = \{pe_{I^c}(F) \mid F \in \mathcal{H}\}^\vee \qquad pe_{\bar{I}^c}(\mathcal{H}^\vee) = \{pe_{\bar{I}^c}(F) \mid F \in \mathcal{H}\}^\vee$$
$$pe_{I^c}(F \to G) = pe_{\bar{I}^c}(F) \to pe_{I^c}(G) \qquad pe_{\bar{I}^c}(F \to G) = pe_{I^c}(F) \to pe_{\bar{I}^c}(G)$$

where a is an atom, $\mathcal{H}$ a set of formulas, and F and G are formulas.

The partial evaluation of an $\mathcal{F}$-program P w.r.t. a two-valued interpretation I^c is $pe_{I^c}(P) = \{pe_{I^c}(r) \mid r \in P\}$ where $pe_{I^c}(r) = H(h) \leftarrow pe_{I^c}(B(r))$. Accordingly, the partial evaluation of rules boils down to replacing satisfied positive occurrences of atoms in rule bodies by $\top$.

We observe the following relationship between the relative one-step operators and partial evaluations.

Observation 19. *Let P be a positive $\mathcal{F}$-program and I^c be a two-valued interpretation.*

Then, we have for any two-valued interpretation I that

$$T_P^{I^c}(I) = T_{pe_{I^c}(P)}(I).$$

Clearly, $pe_{I^c}(P)$ is positive whenever P is positive. In this case, we obtain that $T_P^{I^c}$ is monotone and has a least fixed point corresponding to the least model of $pe_{I^c}(P)$.

We use this correspondence to define a contextual stable operator.

Definition 4. Let P be an $\mathcal{F}$-program and I^c be a two-valued interpretation.

For any two-valued interpretation J, we define the ID-*stable operator relative* to I^c as

$$S_P^{I^c}(J) = LM(pe_{I^c}(P)_J).$$

While the operator is antimonotone w.r.t. its argument J, it is monotone regarding its parameter I^c. Note that $pe_{I^c}(P)_J = pe_{I^c}(P_J)$.

Proposition 20. *Let P be an $\mathcal{F}$-program, and I^c and J be two-valued interpretations.*

We get the following properties:

(a) *$J' \subseteq J$ implies $S_P^{I^c}(J) \subseteq S_P^{I^c}(J')$, and*
(b) *$I^{c\prime} \subseteq I^c$ implies $S_P^{I^{c\prime}}(J) \subseteq S_P^{I^c}(J)$.*

See proof on page 73.

Moreover, we observe the following properties.

Observation 21. *Let P be an $\mathcal{F}$-program, and I^c and J be two-valued interpretations.*

We get the following properties:

(a) $S_P^{\emptyset}(J) = S_P(J)$,

(b) $S_P^{I^c}(J) \subseteq H(P)$, *and*
(c) $S_P^{I^c}(J) = S_P^{I^c \cap B(P)^+}(J \cap B(P)^-)$.

By building on the relative stable operator, we next define its well-founded counterpart. Unlike above, the context is now captured by a four-valued interpretation.

Definition 5. Let P be an $\mathcal{F}$-program and (I^c, J^c) be a four-valued interpretation.

For any four-valued interpretation (I, J), we define the *well-founded operator relative* to (I^c, J^c) as

$$W_P^{(I^c,J^c)}(I, J) = (S_P^{I^c}(J \cup J^c), S_P^{J^c}(I \cup I^c)).$$

As above, we drop parentheses and simply write $W_P^{I,J}$ instead of $W_P^{(I,J)}$. Also, we keep refraining from prepending the prefix 'ID' to the well-founded operator along with all concepts derived from it below.

Unlike the stable operator, the relative well-founded one is monotone on both its argument and parameter.

Proposition 22. *Let P be an $\mathcal{F}$-program, and (I, J) and (I^c, J^c) be four-valued interpretations.*

We get the following properties:
(a) $(I', J') \leq_p (I, J)$ *implies* $W_P^{I^c,J^c}(I', J') \leq_p W_P^{I^c,J^c}(I, J)$, *and*
(b) $(I^{c\prime}, J^{c\prime}) \leq_p (I^c, J^c)$ *implies* $W_P^{I^{c\prime},J^{c\prime}}(I, J) \leq_p W_P^{I^c,J^c}(I, J)$.

See proof on page 74.

From Proposition 22 (a) and Theorem 1 (a), we get that the relative well-founded operator has a least fixed point.

Definition 6. Let P be an $\mathcal{F}$-program and (I^c, J^c) be a four-valued interpretation.

We define the *well-founded model* of P *relative* to (I^c, J^c), written $WM^{(I^c,J^c)}(P)$, as the least fixed point of $W_P^{I^c,J^c}$.

Whenever clear from context, we keep dropping parentheses and simply write $WM^{I,J}(P)$ instead of $WM^{(I,J)}(P)$.

In what follows, we use the relativized concepts defined above to delineate the semantics and resulting simplifications of the sequence of subprograms resulting from a grounder's decomposition of the original program. For simplicity, we first present two propositions capturing the composition under stable and well-founded operations, before we give the general case involving a sequence of programs.

Just like superscript ^{c}, we use the superscript ^{e} (and similarly letter E further below) to indicate atoms whose defining rules are yet to come.

As in traditional splitting, we begin by differentiating a bottom and a top program. In addition to the input atoms J and context atoms in I^c, we moreover distinguish a set of external atoms, I^e, which occur in the bottom

program but are defined in the top program. Accordingly, the bottom program has to be evaluated relative to $I^{\mathrm{c}} \cup I^{\mathrm{e}}$ (and not just I^{c} as above) to consider what could be derived by the top program.

Proposition 23. *Let P^{b} and P^{t} be $\mathcal{F}$-programs, I^{c} and J be two-valued interpretations, $I = S^{I^{\mathrm{c}}}_{P^{\mathrm{b}} \cup P^{\mathrm{t}}}(J)$, $I^{\mathrm{e}} = I \cap (B(P^{\mathrm{b}})^{+} \cap H(P^{\mathrm{t}}))$, $I^{\mathrm{b}} = S^{I^{\mathrm{c}} \cup I^{\mathrm{e}}}_{P^{\mathrm{b}}}(J)$, and $I^{\mathrm{t}} = S^{I^{\mathrm{c}} \cup I^{\mathrm{b}}}_{P^{\mathrm{t}}}(J)$.*

Then, we have $I = I^{\mathrm{b}} \cup I^{\mathrm{t}}$.

See proof on page 74.

For characterizing the relative well-founded models of split programs, we use four-valued interpretations, $(I^{\mathrm{c}}, J^{\mathrm{c}})$ and $(I^{\mathrm{e}}, J^{\mathrm{e}})$, to capture context and external atoms, respectively.

Proposition 24. *Let P^{b} and P^{t} be $\mathcal{F}$-programs, $(I^{\mathrm{c}}, J^{\mathrm{c}})$ be a four-valued interpretation, $(I, J) = WM^{I^{\mathrm{c}}, J^{\mathrm{c}}}(P^{\mathrm{b}} \cup P^{\mathrm{t}})$, $(I^{\mathrm{e}}, J^{\mathrm{e}}) = (I, J) \sqcap (B(P^{\mathrm{b}})^{\pm} \cap H(P^{\mathrm{t}}))$, $(I^{\mathrm{b}}, J^{\mathrm{b}}) = WM^{(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}})}(P^{\mathrm{b}})$, and $(I^{\mathrm{t}}, J^{\mathrm{t}}) = WM^{(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{b}}, J^{\mathrm{b}})}(P^{\mathrm{t}})$.*

Then, we have $(I, J) = (I^{\mathrm{b}}, J^{\mathrm{b}}) \sqcup (I^{\mathrm{t}}, J^{\mathrm{t}})$.

See proof on page 75.

Partially expanding the statements of the two previous results nicely reflects the decomposition of the application of the ID-stable operator and the well-founded founded model of a program:

$$S^{I^{\mathrm{c}}}_{P^{\mathrm{b}} \cup P^{\mathrm{t}}}(J) = S^{I^{\mathrm{c}} \cup I^{\mathrm{e}}}_{P^{\mathrm{b}}}(J) \cup S^{I^{\mathrm{c}} \cup I^{\mathrm{b}}}_{P^{\mathrm{t}}}(J) \text{ and}$$

$$WM^{I^{\mathrm{c}}, J^{\mathrm{c}}}(P^{\mathrm{b}} \cup P^{\mathrm{t}}) = WM^{(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}})}(P^{\mathrm{b}}) \sqcup WM^{(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{b}}, J^{\mathrm{b}})}(P^{\mathrm{t}}).$$

Note that the formulation of both propositions forms the external interpretations, I^{e} and $(I^{\mathrm{e}}, J^{\mathrm{e}})$, by selecting atoms from the overarching interpretation I or the well-founded model (I, J), respectively. This warrants the correspondence of the overall interpretations to the union of the bottom and top interpretations. This global approach is dropped below (after the next example) and leads to less precise composed models.

Example 6. Let us illustrate the above approach via the following program.

(P^{b}) a

(P^{b}) b

(P^{t}) $c \leftarrow a$

(P^{t}) $d \leftarrow \neg b$

The well-founded model of this program relative to $(I^{\mathrm{c}}, J^{\mathrm{c}}) = (\emptyset, \emptyset)$ is

$$(I, J) = (\{a, b, c\}, \{a, b, c\}).$$

First, we partition the four rules of the program into P^{b} and P^{t} as given above. We get $(I^{\mathrm{e}}, J^{\mathrm{e}}) = (\emptyset, \emptyset)$ since $B(P^{\mathrm{b}})^{\pm} \cap H(P^{\mathrm{t}}) = \emptyset$. Let us evaluate P^{b} before P^{t}. The well-founded model of P^{b} relative to $(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}})$ is

$$(I^{\mathrm{b}}, J^{\mathrm{b}}) = (\{a, b\}, \{a, b\}).$$

With this, we calculate the well-founded model of P^{t} relative to $(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{b}}, J^{\mathrm{b}})$:

$$(I^{\mathrm{t}}, J^{\mathrm{t}}) = (\{c\}, \{c\}).$$

We see that the union of $(I^{\mathrm{b}}, J^{\mathrm{b}}) \sqcup (I^{\mathrm{t}}, J^{\mathrm{t}})$ is the same as the well-founded model of $P^{\mathrm{b}} \cup P^{\mathrm{t}}$ relative to $(I^{\mathrm{c}}, J^{\mathrm{c}})$.

This corresponds to standard splitting in the sense that $\{a, b\}$ is a splitting set for $P^{\mathrm{b}} \cup P^{\mathrm{t}}$ and P^{b} is the "bottom" and P^{t} is the "top" (cf. [48]).

For a complement, let us reverse the roles of P^{b} and P^{t}. Unlike above, body atoms in P^{b} now occur in rule heads of P^{t}, i.e., $B(P^{\mathrm{b}})^{\pm} \cap H(P^{\mathrm{t}}) = \{a, b\}$. We thus get $(I^{\mathrm{e}}, J^{\mathrm{e}}) = (\{a, b\}, \{a, b\})$. The well-founded model of P^{b} relative to $(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}})$ is

$$(I^{\mathrm{b}}, J^{\mathrm{b}}) = (\{c\}, \{c\}).$$

And the well-founded model of P^{t} relative to $(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{b}}, J^{\mathrm{b}})$ is

$$(I^{\mathrm{t}}, J^{\mathrm{t}}) = (\{a, b\}, \{a, b\}).$$

Again, we see that the union of both models is identical to (I, J).

This decomposition has no direct correspondence to standard splitting.

Next, we generalize the previous results from two programs to sequences of programs. For this, we let $\mathbb{I}$ be a well-ordered index set and direct our attention to sequences $(P_i)_{i \in \mathbb{I}}$ of $\mathcal{F}$-programs.

Definition 7. Let $(P_i)_{i \in \mathbb{I}}$ be a sequence of $\mathcal{F}$-programs.

We define the *well-founded model* of $(P_i)_{i \in \mathbb{I}}$ as

$$WM((P_i)_{i \in \mathbb{I}}) = \bigsqcup_{i \in \mathbb{I}} (I_i, J_i) \tag{2}$$

where

$$E_i = B(P_i)^{\pm} \cap \bigcup_{i<j} H(P_j), \tag{3}$$

$$(I_i^{\mathrm{c}}, J_i^{\mathrm{c}}) = \bigsqcup_{j<i} (I_j, J_j), \text{ and} \tag{4}$$

$$(I_i, J_i) = WM^{(I_i^{\mathrm{c}}, J_i^{\mathrm{c}}) \sqcup (\emptyset, E_i)}(P_i). \tag{5}$$

The well-founded model of a program sequence is itself assembled in (2) from a sequence of well-founded models of the individual subprograms in (5). This provides us with semantic guidance for successive program simplification, as shown below. In fact, proceeding along the sequence of subprograms reflects the iterative approach of a grounding algorithm, one component is grounded at a time. At each stage $i \in \mathbb{I}$, this takes into account the truth values of atoms instantiated in previous iterations, viz. $(I_i^{\mathrm{c}}, J_i^{\mathrm{c}})$, as well as dependencies to upcoming components in E_i. Note that unlike Proposition 24, the external atoms in E_i are identified purely syntactically, and the interpretation $(\emptyset, E_i)$ treats them as unknown. Grounding is thus

performed under incomplete information and each well-founded model in (5) can be regarded as an over-approximation of the actual one. This is enabled by the monotonicity of the well-founded operator in Proposition 22 (b) that only leads to a less precise result when overestimating its parameter, as made precise in the following theorem.

Theorem 25. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{F}$-programs.*
Then, $WM((P_i)_{i\in\mathbb{I}}) \leq_p WM(\bigcup_{i\in\mathbb{I}} P_i)$.

See proof on page 79.

In fact, no loss is encountered when head literals never occur in the bodies of previous programs, as shown next.

Corollary 26. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{F}$-programs and E_i defined as in (3).*
If $E_i = \emptyset$ for all $i \in \mathbb{I}$ then $WM((P_i)_{i\in\mathbb{I}}) = WM(\bigcup_{i\in\mathbb{I}} P_i)$.

See proof on page 79.

Whenever head atoms do not interfere with negative body literals, the relative well founded-model of a program can be calculated with just two applications of the relative ID-stable model operator.

Lemma 27. *Let P be an $\mathcal{F}$-program such that $B(P)^- \cap H(P) = \emptyset$ and (I^c, J^c) be a four-valued interpretation.*
Then, $WM^{I^c,J^c}(P) = (S_P^{I^c}(J^c), S_P^{J^c}(I^c))$.

See proof on page 80.

Any sequence as in Corollary 26 in which each P_i additionally satisfies the precondition of Lemma 27 has a total well-founded model. Furthermore, the well-founded model of such a sequence can be calculated with just two (independent) applications of the relative ID-stable model operator per program P_i in the sequence.

The next two results transfer Theorem 25 and Corollary 26 to program simplification by successively simplifying programs with the respective well-founded models of the previous programs.

Theorem 28. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{F}$-programs, $(I, J) = WM((P_i)_{i\in\mathbb{I}})$, and E_i, (I_i^c, J_i^c), and (I_i, J_i) be defined as in (3) to (5).*
Then, $P_k^{I,J} \subseteq P_k^{(I_k^c,J_k^c)\sqcup(I_k,J_k)\sqcup(\emptyset,E_k)} \subseteq P_k$ for all $k \in \mathbb{I}$.

See proof on page 80.

Corollary 29. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{F}$-programs, $(I, J) = WM((P_i)_{i\in\mathbb{I}})$, and E_i, (I_i^c, J_i^c), and (I_i, J_i) be defined as in (3) to (5).*
If $E_i = \emptyset$ for all $i \in \mathbb{I}$, then $P_k^{I,J} = P_k^{(I_k^c,J_k^c)\sqcup(I_k,J_k)}$ for all $k \in \mathbb{I}$.

See proof on page 80.

Clearly, the best simplifications are obtained when knowing the actual well-founded model of the overall program. This can be achieved whenever E_i is empty, that is, if there is no need to approximate the impact of upcoming atoms, otherwise we can only guarantee the bounds in Theorem 28.

Corollary 30. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{R}$-programs, and (I, J) be the well-founded model of $\bigcup_{i\in\mathbb{I}} P_i$.*

Then, $\bigcup_{i\in\mathbb{I}} P_i$ and $\bigcup_{i\in\mathbb{I}} Q_i$ with $P_i^{I,J} \subseteq Q_i \subseteq P_i$ have the same well-founded and stable models.

See proof on page 81.

Corollary 31. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{R}$-programs, and E_i, (I_i^c, J_i^c), and (I_i, J_i) be defined as in (3) to (5).*

Then, $\bigcup_{i\in\mathbb{I}} P_i$ and $\bigcup_{i\in\mathbb{I}} P_i^{(I_i^c,J_i^c)\sqcup(I_i,J_i)\sqcup(\emptyset,E_i)}$ have the same well-founded and stable models.

See proof on page 81.

Let us mention that the two previous results extend to sequences of $\mathcal{F}$-programs and their well-founded models but not their stable models.

Let us illustrate the above results with the following two examples.

Example 7. To illustrate Theorem 25, let us consider the following programs, P_1 and P_2:

(P_1) $a \leftarrow \neg c$

(P_1) $b \leftarrow \neg d$

(P_2) $c \leftarrow \neg b$

(P_2) $d \leftarrow e$

The well-founded model of $P_1 \cup P_2$ is

$$(I, J) = (\{a, b\}, \{a, b\}).$$

Let us evaluate P_1 before P_2. While no head literals of P_2 occur positively in P_1, the head literals c and d of P_2 occur negatively in rule bodies of P_1. Hence, we get $E_1 = \{c, d\}$ and treat both atoms as unknown while calculating the well-founded model of P_1 relative to $(\emptyset, \{c, d\})$:

$$(I_1, J_1) = (\emptyset, \{a, b\}).$$

We obtain that both a and b are unknown. With this and $E_2 = \emptyset$, we can calculate the well-founded model of P_2 relative to (I_1, J_1):

$$(I_2, J_2) = (\emptyset, \{c\}).$$

We see that because a is unknown, we have to derive c as unknown, too. And because there is no rule defining e, we cannot derive d. Hence, $(I_1, J_1)\sqcup(I_2, J_2)$ is less precise than (I, J) because, when evaluating P_1, it is not yet known that c is true and d is false.

Next, we illustrate the simplified programs according to Theorem 28:

(P_1)	$a \leftarrow \neg c$	$a \leftarrow \neg c$
(P_1)	$b \leftarrow \neg d$	$b \leftarrow \neg d$
(P_2)		$c \leftarrow \neg b$

The left column contains the simplification of $P_1 \cup P_2$ w.r.t. (I, J) and the right column the simplification of P_1 w.r.t. (I_1, J_1) and P_2 w.r.t. $(I_1, J_1) \sqcup (I_2, J_2)$. Note that $d \leftarrow e$ has been removed in both columns because e is false in both (I, J) and $(I_1, J_1) \sqcup (I_2, J_2)$. But we can only remove $c \leftarrow \neg b$ from the left column because, while b is false in (I, J), it is unknown in $(I_1, J_1) \sqcup (I_2, J_2)$.

Example 8. Next, let us illustrate Corollary 26 on an example. We take the same rules as in Example 7 but use a different sequence:

$$(P_1) \qquad d \leftarrow e$$

$$(P_1) \qquad b \leftarrow \neg d$$

$$(P_2) \qquad c \leftarrow \neg b$$

$$(P_2) \qquad a \leftarrow \neg c$$

Observe that the head literals of P_2 do not occur in the bodies of P_1, i.e., $E_1 = B(P_1)^{\pm} \cap H(P_2) = \emptyset$. The well-founded model of P_1 is

$$(I_1, J_1) = (\{b\}, \{b\}).$$

And the well-founded model of P_2 relative to $(\{b\}, \{b\})$ is

$$(I_2, J_2) = (\{a\}, \{a\}).$$

Hence, the union of both models is identical to the well-founded model of $P_1 \cup P_2$.

Next, we investigate the simplified program according to Corollary 29:

$$(P_1) \qquad b \leftarrow \neg d$$

$$(P_2) \qquad a \leftarrow \neg c$$

As in Example 7, we delete rule $d \leftarrow e$ because e is false in (I_1, J_1). But this time, we can also remove rule $c \leftarrow \neg b$ because b is true in $(I_1, J_1) \sqcup (I_2, J_2)$.

4. Aggregate Programs

We now turn to programs with aggregates and, at the same time, to programs with variables. That is, we now deal with finite nonground programs that may be turned by instantiation into infinite ground programs. Our concepts follow the ones in [24]; the semantics of aggregates is aligned with [19] yet lifted to infinitary formulas (cf. [37, 60]).

We consider a *signature* $\Sigma = (F, P, V)$ consisting of sets of function, predicate, and variables symbols. The sets of variable and function symbols are disjoint. *Function* and *predicate symbols* are associated with non-negative arities. In the following, we use lower case strings for function and predicate symbols, and upper case strings for variable symbols. Also, we often drop the term 'symbol' and simply speak of functions, predicates, and variables.

As usual, *terms* over Σ are defined inductively as follows:

- $v \in V$ is a term and
- $f(t_1, \ldots, t_n)$ is a term if $f \in F$ is a function symbol of arity n and each t_i is a term over Σ.

Parentheses for terms over function symbols of arity 0 are omitted.

Unless stated otherwise, we assume that the set of (zero-ary) functions includes a set of numeral symbols being in a one-to-one correspondence to the integers. For simplicity, we drop this distinction and identify numerals with the respective integers.

An *atom* over signature Σ has form $p(t_1, \ldots, t_n)$ where $p \in P$ is a predicate symbol of arity n and each t_i is a term over Σ. As above, parentheses for atoms over predicate symbols of arity 0 are omitted. Given an atom a over Σ, a *literal* over Σ is either the atom itself or its negation $\neg a$. A literal without negation is called *positive*, and *negative* otherwise.

A *comparison* over Σ has form

$$t_1 \prec t_2 \tag{6}$$

where t_1 and t_2 are terms over Σ and $\prec$ is a relation symbol among $<$, $\leq$, $>$, $\geq$, $=$, and $\neq$.

An *aggregate element* over Σ has form

$$t_1, \ldots, t_m : a_1 \wedge \cdots \wedge a_n \tag{7}$$

where t_i is a term and a_j is an atom, both over Σ for $0 \leq i \leq m$ and $0 \leq j \leq n$. The terms $t_1, \ldots, t_m$ are seen as a tuple, which is empty for $m = 0$. For an aggregate element e of form (7), we use $H(e) = (t_1, \ldots, t_m)$ and $B(e) = \{a_1, \ldots, a_n\}$. We extend both to sets of aggregate elements in the straightforward way, that is, $H(E) = \{H(e) \mid e \in E\}$ and $B(E) = \{B(e) \mid e \in E\}$.

An *aggregate atom* over Σ has form

$$f\{e_1, \ldots, e_n\} \prec s \tag{8}$$

where $n \geq 0$, f is an aggregate name among #count, #sum, #sum$^+$, and #sum$^-$, each e_i is an aggregate element, $\prec$ is a relation symbol among $<$, $\leq$, $>$, $\geq$, $=$, and $\neq$ (as above), and s is a term.

Without loss of generality, we refrain from introducing negated aggregate atoms.[4] We often refer to aggregate atoms simply as aggregates.

An *aggregate program* over Σ is a finite set of *aggregate rules* of form

$$h \leftarrow b_1 \wedge \cdots \wedge b_n$$

where $n \geq 0$, h is an atom over Σ and each b_i is either a literal, a comparison, or an aggregate over Σ. We refer to $b_1, \ldots, b_n$ as body literals, and extend functions $H(r)$ and $B(r)$ to any aggregate rule r.

Examples of aggregate rules are given in (1) and Table 1.

We say that an aggregate rule r is *normal* if its body does not contain aggregates. An aggregate program is normal if all its rules are normal.

[4]Grounders like *lparse* and *gringo* replace aggregates systematically by auxiliary atoms and place them in the body of new rules implying the respective auxiliary atom. This results in programs without occurrences of negated aggregates.

A term, literal, aggregate element, aggregate, rule, or program is *ground* whenever it does not contain any variables.

We assume that all ground terms are totally ordered by a relation $\leq$, which is used to define the relations $<$, $>$, $\geq$, $=$, and $\neq$ in the standard way. For ground terms t_1, t_2 and a corresponding relation symbol $\prec$, we say that $\prec$ holds between t_1 and t_2 whenever the corresponding relation holds between t_1 and t_2. Furthermore, $>$, $\geq$, and $\neq$ hold between ∞ and any other term, and $<$, $\leq$, and $\neq$ hold between $-\infty$ and any other term. Finally, we require that integers are ordered as usual, and that all terms involving function symbols are somehow totally ordered and larger than integers.

For defining sum-based aggregates, we define for a tuple $t = t_1, \ldots, t_m$ of ground terms the following weight functions:

$$w(t) = \begin{cases} t_1 & \text{if } m > 0 \text{ and } t_1 \text{ is an integer} \\ 0 & \text{otherwise,} \end{cases}$$
$$w^+(t) = \max\{w(t), 0\}, \text{ and}$$
$$w^-(t) = \min\{w(t), 0\}.$$

With this at hand, we now define how to apply aggregate functions to sets of tuples of ground terms in analogy to [24].

Definition 8. Let T be a set of tuples of ground terms.

We define

$$\#\text{count}(T) = \begin{cases} |T| & \text{if } T \text{ is finite,} \\ \infty & \text{otherwise,} \end{cases}^5$$
$$\#\text{sum}(T) = \begin{cases} \Sigma_{t \in T} w(t) & \text{if } \{t \in T \mid w(t) \neq 0\} \text{ is finite,} \\ 0 & \text{otherwise,} \end{cases}$$
$$\#\text{sum}^+(T) = \begin{cases} \Sigma_{t \in T} w^+(t) & \text{if } \{t \in T \mid w(t) > 0\} \text{ is finite,} \\ \infty & \text{otherwise, and} \end{cases}$$
$$\#\text{sum}^-(T) = \begin{cases} \Sigma_{t \in T} w^-(t) & \text{if } \{t \in T \mid w(t) < 0\} \text{ is finite,} \\ -\infty & \text{otherwise.} \end{cases}$$

Note that in our setting the application of aggregate functions to infinite sets of ground terms is of theoretical relevance only, since we aim at reducing them to their finite equivalents so that they can be evaluated by a grounder.

A variable is *global* in

- a literal if it occurs in the literal,
- a comparison if it occurs in the comparison,
- an aggregate if it occurs in its bound, and
- a rule if it is global in its head atom or in one of its body literals.

[5] We present two cases in analogy to the subsequent definitions.

For example, in Table 1 variables X and Y are global in the rule, while Z and S are neither global in the rule nor the aggregate.

Definition 9. Let r be an aggregate rule.

We define r to be *safe*

- if all its global variables occur in some positive literal in the body of r and
- if all its non-global variables occurring in an aggregate element e of an aggregate in the body of r, also occur in some positive literal in the condition of e.

For instance, the rule in Table 1 is safe.

Note that comparisons are disregarded in the definition of safety. That is, variables in comparisons have to occur in positive body literals.

An aggregate program is safe if all its rules are safe.

An *instance* of an aggregate rule r is obtained by substituting ground terms for all its global variables. We use $\mathrm{Inst}(r)$ to denote the set of all instances of r and $\mathrm{Inst}(P)$ to denote the set of all ground instances of rules in aggregate program P. An *instance* of an aggregate element e is obtained by substituting ground terms for all its variables. We let $\mathrm{Inst}(E)$ stand for all instances of aggregate elements in a set E. Note that $\mathrm{Inst}(E)$ consists of ground expressions, which is not necessarily the case for $\mathrm{Inst}(r)$.

A literal, aggregate element, aggregate, or rule is *closed* if it does not contain any global variables.

For example, the following rule is an instance of the one in Table 1.

$$\begin{aligned} \mathit{controls}(c_1, c_2) \leftarrow \#\mathrm{sum}^+\{S : \mathit{owns}(c_1, c_2, S); \\ S, Z : \mathit{controls}(c_1, Z), \mathit{owns}(Z, c_2, S)\} > 50 \\ \wedge\ \mathit{company}(c_1) \wedge \mathit{company}(c_2) \wedge c_1 \neq c_2 \end{aligned}$$

Note that both the rule and its aggregate are closed. It is also noteworthy to realize that the two elements of the aggregate induce an infinite set of instances, among them

$$\begin{aligned} 20 &: \mathit{owns}(c_1, c_2, 20) && \text{and} \\ 35, c_2 &: \mathit{controls}(c_1, c_2), \mathit{owns}(c_2, c_3, 35). \end{aligned}$$

We now turn to the semantics of aggregates as introduced in [19] but follow its adaptation to closed aggregates in [24]: Let a be a closed aggregate of form (8) and E be its set of aggregate elements. We say that a set $D \subseteq \mathrm{Inst}(E)$ of its elements' instances *justifies* a, written $D \triangleright a$, if $f(H(D)) \prec s$ holds.

An aggregate a is *monotone* whenever $D_1 \triangleright a$ implies $D_2 \triangleright a$ for all $D_1 \subseteq D_2 \subseteq \mathrm{Inst}(E)$, and accordingly a is *antimonotone* if $D_2 \triangleright a$ implies $D_1 \triangleright a$ for all $D_1 \subseteq D_2 \subseteq \mathrm{Inst}(E)$.

We observe the following monotonicity properties.

Proposition 32 ([37])**.**

- *Aggregates over functions* $\#\text{sum}^+$ *and* $\#\text{count}$ *together with aggregate relations* $>$ *and* $\geq$ *are monotone.*
- *Aggregates over functions* $\#\text{sum}^+$ *and* $\#\text{count}$ *together with aggregate relations* $<$ *and* $\leq$ *are antimonotone.*
- *Aggregates over function* $\#\text{sum}^-$ *have the same monotonicity properties as* $\#\text{sum}^+$ *aggregates with the complementary relation.*

Next, we give the translation τ from aggregate programs to $\mathcal{R}$-programs, derived from [19, 37]:

For a closed literal l, we have

$$\tau(l) = l,$$

for a closed comparison l of form (6), we have

$$\tau(l) = \begin{cases} \top & \text{if } \prec \text{ holds between } t_1 \text{ and } t_2 \\ \bot & \text{otherwise} \end{cases}$$

and for a set L of closed literals, comparisons and aggregates, we have

$$\tau(L) = \{\tau(l) \mid l \in L\}.$$

For a closed aggregate a of form (8) and its set E of aggregate elements, we have

$$\tau(a) = \{\tau(D)^\wedge \to \tau_a(D)^\vee \mid D \subseteq \text{Inst}(E), D \not\models a\}^\wedge \tag{9}$$

where

$$\begin{aligned} \tau_a(D) &= \tau(\text{Inst}(E) \setminus D) \text{ for } D \subseteq \text{Inst}(E), \\ \tau(D) &= \{\tau(e) \mid e \in D\} \text{ for } D \subseteq \text{Inst}(E), \text{ and} \\ \tau(e) &= \tau(B(e))^\wedge \text{ for } e \in \text{Inst}(E). \end{aligned}$$

For a closed aggregate rule r, we have

$$\tau(r) = \tau(H(r)) \leftarrow \tau(B(r))^\wedge.$$

For an aggregate program P, we have

$$\tau(P) = \{\tau(r) \mid r \in \text{Inst}(P)\}. \tag{10}$$

Observe that $\tau(P)$ is indeed an $\mathcal{R}$-program. In fact, only the translation of aggregates introduces $\mathcal{R}$-formulas; rules without aggregates form $\mathcal{N}$-programs.

Example 9. To illustrate Ferraris' approach to the semantics of aggregates, consider a count aggregate a of form

$$\#\text{count}\{X : p(X)\} \geq n.$$

Since the aggregate is non-ground, the set G of its element's instances consists of all $t : p(t)$ for each ground term t.

The count aggregate cannot be justified by any subset D of G satisfying $|\{t \mid t : p(t) \in D\}| < n$, or $D \not\models a$ for short. Accordingly, we have that $\tau(a)$ is the conjunction of all formulas

$$\{p(t) \mid t : p(t) \in D\}^{\wedge} \rightarrow \{p(t) \mid t : p(t) \in (G \setminus D)\}^{\vee} \tag{11}$$

such that $D \subseteq G$ and $D \not\models a$. Restricting the set of ground terms to the numerals $1, 2, 3$ and letting $n = 2$ results in the formulas

$$\begin{aligned} \top &\rightarrow p(1) \vee p(2) \vee p(3), \\ p(1) &\rightarrow p(2) \vee p(3), \\ p(2) &\rightarrow p(1) \vee p(3), \text{ and} \\ p(3) &\rightarrow p(1) \vee p(2). \end{aligned}$$

Note that a smaller number of ground terms than n yields an unsatisfiable set of formulas.

However, it turns out that a Ferraris-style [19,37] translation of aggregates is too weak for propagating monotone aggregates in our ID-based setting. That is, when propagating possible atoms (i.e., the second component of the well-founded model), an ID-reduct may become satisfiable although the original formula is not. So, we might end up with too many possible atoms and a well-founded model that is not as precise as it could be. To see this, consider the following example.

Example 10. For some $m, n \geq 0$, the program $P_{m,n}$ consists of the following rules:

$$\begin{aligned} p(i) &\leftarrow \neg q(i) && \text{for } 1 \leq i \leq m \\ q(i) &\leftarrow \neg p(i) && \text{for } 1 \leq i \leq m \\ r &\leftarrow \#\text{count}\{X : p(X)\} \geq n \end{aligned}$$

Given the ground instances G of the aggregate's elements and some two-valued interpretation I, observe that

$$\tau(\#\text{count}\{X : p(X)\} \geq n)_I$$

is classically equivalent to

$$\tau(\#\text{count}\{X : p(X)\} \geq n)_I \vee \{p(t) \in B(G) \mid p(t) \notin I\}^{\vee}. \tag{12}$$

Next, observe that for $1 \leq m < n$, the four-valued interpretation $(I, J) = (\emptyset, H(\tau(P_{m,n})))$ is the well-founded model of $P_{m,n}$:

$$\begin{aligned} S_{\tau(P_{m,n})}(J) &= I \text{ and} \\ S_{\tau(P_{m,n})}(I) &= J. \end{aligned}$$

Ideally, atom r should not be among the possible atoms because it can never be in a stable model. Nonetheless, it is due to the second disjunct in (12).

Note that not just monotone aggregates exhibit this problem. In general, we get for a closed aggregate a with elements E and an interpretation I that

$$\tau(a)_I \text{ is classically equivalent to } \tau(a)_I \vee \{c \in B(\mathrm{Inst}(E)) \mid I \not\models c\}^{\vee}.$$

The second disjunct is undesirable when propagating possible atoms.

To address this shortcoming, we augment the aggregate translation so that it provides stronger propagation. The result of the augmented translation is strongly equivalent to that of the original translation (cf. Proposition 33). Thus, even though we get more precise well-founded models, the stable models are still contained in them.

Definition 10. We define τ^* as the translation obtained from τ by replacing the case of closed aggregates in (9) by the following:

For a closed aggregate a of form (8) and its set E of aggregate elements, we have

$$\tau^*(a) = \{\tau(D)^{\wedge} \rightarrow \tau_a^*(D)^{\vee} \mid D \subseteq \mathrm{Inst}(E), D \not\models a\}^{\wedge}$$

where

$$\tau_a^*(D) = \{\tau(C)^{\wedge} \mid C \subseteq \mathrm{Inst}(E) \setminus D, C \cup D \rhd a\} \text{ for } D \subseteq \mathrm{Inst}(E).$$

Note that just as τ also τ^* is recursively applied to the whole program.

Let us illustrate the modified translation by reconsidering the aggregate from Example 9.

Example 11. Let us reconsider the count aggregate a:

$$\#\mathrm{count}\{X : p(X)\} \geq n$$

As with $\tau(a)$ in Example 9, $\tau^*(a)$ yields a conjunction of formulas, one conjunct for each set $D \subseteq \mathrm{Inst}(E)$ satisfying $D \not\models a$ of the form:

(13)
$$\{B(e) \mid e \in D\}^{\wedge} \rightarrow \big\{\{B(e) \mid e \in (C \setminus D)\}^{\wedge} \mid C \rhd a, D \subseteq C \subseteq \mathrm{Inst}(E)\big\}^{\vee}$$

Restricting again the set of ground terms to the numerals $1, 2, 3$ and letting $n = 2$ results now in the formulas

$$\top \rightarrow (p(1) \wedge p(2)) \vee (p(1) \wedge p(3)) \vee (p(2) \wedge p(3)) \vee (p(1) \wedge p(2) \wedge p(3)),$$
$$p(1) \rightarrow p(2) \vee p(3) \vee (p(2) \wedge p(3)),$$
$$p(2) \rightarrow p(1) \vee p(3) \vee (p(1) \wedge p(3)), \text{ and}$$
$$p(3) \rightarrow p(1) \vee p(2) \vee (p(1) \wedge p(2)).$$

Note that the last disjunct can be dropped from each rule's consequent. And as above, a smaller number of ground terms than n yields an unsatisfiable set of formulas.

The next result ensures that $\tau(P)$ and $\tau^*(P)$ have the same stable models for any aggregate program P.

Proposition 33. *Let a be a closed aggregate.*
Then, $\tau(a)$ and $\tau^*(a)$ are strongly equivalent.

See proof on page 81.

Example 12. For illustration, reconsider Program $P_{m,n}$ from Example 10.

As above, we apply the well-founded operator to program $P_{m,n}$ for $m < n$ and four-valued interpretation $(I, J) = (\emptyset, H(\tau^*(P_{m,n})))$:

$$S_{\tau^*(P_{m,n})}(J) = I \text{ and}$$
$$S_{\tau^*(P_{m,n})}(I) = J \setminus \{r\}.$$

Unlike before, r is now found to be false since it does not belong to $S_{\tau^*(P_{m,n})}(\emptyset)$.

To see this, let us anticipate Proposition 34 and observe that $\tau^*(a)_I = \tau^*(a)$ for $a = \#\text{count}\{X : p(X)\} \geq n$ and any interpretation I. Hence, our refined translation τ^* avoids the problematic disjunct in (12) on the right. By Proposition 33, we can use $\tau^*(P_{m,n})$ instead of $\tau(P_{m,n})$; both formulas have the same stable models.

Proposition 34. *Let a be a closed aggregate.*
If a is monotone, then $\tau^*(a)_I$ is classically equivalent to $\tau^*(a)$ for any two-valued interpretation I.

See proof on page 82.

Note that $\tau^*(a)$ is a negative formula whenever a is antimonotone; cf. Proposition 51.

Using this proposition, we augment the translation τ^* to replace monotone aggregates a by the strictly positive formula $\tau^*(a)_\emptyset$. That is, we only keep the implication with the trivially true antecedent in the aggregate translation.

While τ^* improves on propagation, it may still produce infinitary $\mathcal{R}$-formulas when applied to aggregates. This issue is addressed by restricting the translation to a set of (possible) atoms.

Definition 11. Let J be a two-valued interpretation. We define the translation τ^*_J as the one obtained from τ by replacing the case of closed aggregates in (9) by the following:

For a closed aggregate a of form (8) and its set E of aggregate elements, we have

$$\tau^*_J(a) = \{\tau(D)^\wedge \to \tau^*_{a,J}(D)^\vee \mid D \subseteq \text{Inst}(E)|_J, D \not\models a\}^\wedge$$

where

$$\tau^*_{a,J}(D) = \{\tau(C)^\wedge \mid C \subseteq \text{Inst}(E)|_J \setminus D, C \cup D \triangleright a\} \text{ and}$$
$$\text{Inst}(E)|_J = \{e \in \text{Inst}(E) \mid B(e) \subseteq J\}.$$

Note that $\tau^*_J(a)$ is a finite formula whenever J is finite.

Clearly, τ^*_J also conforms to τ^* except for the restricted translation for aggregates defined above. The next proposition elaborates this by showing that τ^*_J and τ^* behave alike whenever J limits the set of possible atoms.

Proposition 35. *Let a be a closed aggregate, and $I \subseteq J$ and $X \subseteq J$ be two-valued interpretations.*

Then,

(a) $X \models \tau^*(a)$ *iff* $X \models \tau^*_J(a)$,
(b) $X \models \tau^*(a)_I$ *iff* $X \models \tau^*_J(a)_I$, *and*
(c) $X \models \tau^*(a)^I$ *iff* $X \models \tau^*_J(a)^I$.

See proof on page 82.

We are now in a position to outline how safe (non-ground) aggregate programs can be turned into equivalent finite ground programs consisting of finite formulas only.

To this end, consider a safe aggregate program P along with the well-founded model (I, J) of $\tau^*(P)$. We have already seen in Section 2.5 that $\tau^*(P)$ and $\tau^*(P)^{I,J}$ have the same stable models, just like $\tau^*(P)^{I,J}$ and $\tau^*_J(P)^{I,J}$ in view of Proposition 35.

Now, if (I, J) is finite, then $\tau^*(P)^{I,J}$ is finite, too. The safety of all rules in P implies that all global variables appear in positive body literals. Thus, the number of ground instances of each rule in $\tau^*(P)^{I,J}$ is determined by the number of possible substitutions for its global variables. Clearly, there are only finitely many possible substitutions such that all positive body literals are satisfied by a finite interpretation J (cf. Definition 1). Furthermore, if J is finite, aggregate translations in $\tau^*_J(P)^{I,J}$ introduce finite subformulas only. Thus, in this case, we obtain a finite propositional formula that has the same stable models as $\tau^*(P)$ (as well as $\tau(P)$, the traditional Ferraris-style semantics of P [19, 37]).

An example of a class of aggregate programs inducing finite well-founded models as above consists of programs over a signature with nullary function symbols only. Any such program can be turned into an equivalent finite set of finite propositional rules.

Example 13. Let $P_{m,n}$ be the program from Example 10. The well-founded model (I, J) of $\tau^*(P_{m,n})$ is $(\emptyset, H(\tau^*(P_{m,n})))$ if $n \leq m$.

The translation $\tau^*_J(P_{3,2})$ consists of the rules

$$\begin{aligned} p(1) &\leftarrow \neg q(1), & q(1) &\leftarrow \neg p(1), \\ p(2) &\leftarrow \neg q(2), & q(2) &\leftarrow \neg p(2), \\ p(3) &\leftarrow \neg q(3), & q(3) &\leftarrow \neg p(3), \text{ and} \\ r &\leftarrow \tau^*_J(\#\text{count}\{X : p(X)\} \geq 2) \end{aligned}$$

where the aggregate translation corresponds to the conjunction of the formulas in Example 11. Note that the translation $\tau^*_J(P_{3,2})$ is independent of the signature of $P_{3,2}$; any compatible signature including all numerals can be chosen.

5. Dependency Analysis

We now further refine our semantic approach to reflect actual grounding processes. In fact, modern grounders process programs on-the-fly by grounding one rule after another without storing any rules. At the same time, they try to determine certain, possible, and false atoms. Unfortunately, well-founded models cannot be computed on-the-fly, which is why we define the concept of an approximate model. More precisely, we start by defining instantiation sequences of (non-ground) aggregate programs based on their rule dependencies. We show that approximate models of instantiation sequences are underapproximations of the well-founded model of the corresponding sequence of (ground) $\mathcal{R}$-programs, as defined in Section 3. The precision of both types of models coincides on stratified programs. We illustrate our concepts comprehensively at the end of this section in Examples 14 and 15.

To begin with, we extend the notion of positive and negative literals to aggregate programs. For atoms a, we define $a^+ = (\neg a)^- = \{a\}$ and $a^- = (\neg a)^+ = \emptyset$. For comparisons a, we define $a^+ = a^- = \emptyset$. For aggregates a with elements E, we define positive and negative atom occurrences, using Proposition 34 to refine the case for monotone aggregates:

- $a^+ = \bigcup_{e \in E} B(e)$,
- $a^- = \emptyset$ if a is monotone, and
- $a^- = \bigcup_{e \in E} B(e)$ if a is not monotone.

For a set of body literals B, we define $B^+ = \bigcup_{b \in B} b^+$ and $B^- = \bigcup_{b \in B} b^-$.

We see in the following, that a special treatment of monotone aggregates yields better approximations of well-founded models. A similar case could be made for antimonotone aggregates but had led to a more involved algorithmic treatment.

Inter-rule dependencies are determined via the predicates appearing in their heads and bodies. We define $\mathrm{pred}(a)$ to be the predicate symbol associated with atom a and $\mathrm{pred}(A) = \{\mathrm{pred}(a) \mid a \in A\}$ for a set A of atoms. An aggregate rule r_1 *depends* on another aggregate rule r_2 if $\mathrm{pred}(H(r_2)) \in \mathrm{pred}(B(r_1)^{\pm})$. Rule r_1 depends *positively* or *negatively* on r_2 if $\mathrm{pred}(H(r_2)) \in \mathrm{pred}(B(r_1)^+)$ or $\mathrm{pred}(H(r_2)) \in \mathrm{pred}(B(r_2)^-)$, respectively.

The *strongly connected components* of an aggregate program P are the equivalence classes under the transitive closure of the dependency relation between all rules in P. A strongly connected component P_1 *depends* on another strongly connected component P_2 if there is a rule in P_1 that depends on some rule in P_2. The transitive closure of this relation is anti-symmetric.

A strongly connected component of an aggregate program is *unstratified* if it depends negatively on itself or if it depends on an unstratified component. A component is *stratified* if it is not unstratified.

Just for the record, we summarize next how dependencies transfer from non-ground aggregate programs to the corresponding ground $\mathcal{R}$-programs.

Observation 36. *Let P_1 and P_2 be aggregate programs, and $G_1 = \tau^*(P_1)$ and $G_2 = \tau^*(P_2)$.*

Then,

(a) *P_1 does not depend on P_2 implies $B(G_1)^{\pm} \cap H(G_2) = \emptyset$,*
(b) *P_1 does not positively depend on P_2 implies $B(G_1)^{+} \cap H(G_2) = \emptyset$,*
(c) *P_1 does not negatively depend on P_2 implies $B(G_1)^{-} \cap H(G_2) = \emptyset$.*

A topological ordering of the strongly connected components is then used to guide grounding.

Definition 12. We define an *instantiation sequence* for P as a sequence $(P_i)_{i \in \mathbb{I}}$ of its strongly connected components such that $i < j$ if P_j depends on P_i.

Note that the components can always be well-ordered because we only consider finite aggregate programs. Examples of (refined) instantiation sequences are given in Figures 1 and 2.

Before further refining instantiation sequences below, we pin down two properties of interest. First of all, there are no external atoms in the components of instantiation sequences.

Lemma 37. *Let P be an aggregate program and $(P_i)_{i \in \mathbb{I}}$ be an instantiation sequence for P.*

Then, for the sequence $(G_i)_{i \in \mathbb{I}}$ with $G_i = \tau^*(P_i)$, we have $E_i = \emptyset$ for each $i \in \mathbb{I}$ where E_i is defined as in (3).

See proof on page 83.

Together with Corollary 26, this shows that the consecutive construction of the well-founded model along an instantiation sequence results in the well-founded model of the entire program.

Moreover, for each stratified component in an instantiation sequence, we obtain a total well-founded model.

Lemma 38. *Let P be an aggregate program and $(P_i)_{i \in \mathbb{I}}$ be an instantiation sequence for P.*

Then, for the sequence $(G_i)_{i \in \mathbb{I}}$ with $G_i = \tau^*(P_i)$, we have $I_i = J_i = S_{G_i}^{I_i^c}(I_i^c)$ for each stratified component P_i where (I_i^c, J_i^c) and (I_i, J_i) are defined as in (4) and (5) in the construction of the well-founded model of $(G_i)_{i \in \mathbb{I}}$ in Definition 7.

See proof on page 83.

Note that the total well-founded model of each stratified component can be computed with one application of the ID-stable operator.

Trivial examples of such stratified components are P_1 to P_4 in Example 14 and P_1 to P_8 in Example 15, all of which consist of facts only and thus yield total well-founded models. This also applies to P_9 in Example 15 because it only involves positive dependencies. Since this includes all components of the Company Controls instance from Section 1, it gets completely evaluated during grounding.

We further refine instantiation sequences by partitioning each component along its positive dependencies.

Definition 13. Let P be an aggregate program and $(P_i)_{i\in\mathbb{I}}$ be an instantiation sequence for P. Furthermore, for each $i \in \mathbb{I}$, let $(P_{i,j})_{j\in\mathbb{I}_i}$ be an instantiation sequence of P_i considering positive dependencies only.

A *refined instantiation sequence* for P is a sequence $(P_{i,j})_{(i,j)\in\mathbb{J}}$ where the index set $\mathbb{J} = \{(i,j) \mid i \in \mathbb{I}, j \in \mathbb{I}_i\}$ is ordered lexicographically.

We call $(P_{i,j})_{(i,j)\in\mathbb{J}}$ a refinement of $(P_i)_{i\in\mathbb{I}}$.

We define a component $P_{i,j}$ to be *stratified* or *unstratified* if the encompassing component P_i is stratified or unstratified, respectively.

Examples of refined instantiation sequences are given in Figures 1 and 2.

The advantage of such refinements is that they yield better or equal approximations (cf. Theorem 41 and Example 14). On the downside, they admit once more external atoms (cf. Lemma 37), although their scope is limited to negative body literals:

Lemma 39. *Let P be an aggregate program and $(P_{i,j})_{(i,j)\in\mathbb{J}}$ be a refined instantiation sequence for P.*

Then, for the sequence $(G_{i,j})_{(i,j)\in\mathbb{J}}$ with $G_{i,j} = \tau^*(P_{i,j})$, we have $E_{i,j} \cap B(G_{i,j})^+ = \emptyset$ for each $(i,j) \in \mathbb{J}$ where $E_{i,j}$ is defined as in (3).

See proof on page 83.

We have already seen in Section 3 that external atoms may lead to less precise semantic characterizations. This is just the same in the non-ground case, whenever a component comprises predicates that are defined in a following component of a refined instantiation sequence. This leads us to the concept of an approximate model obtained by overapproximating the extension of such externally defined predicates.

Definition 14. Let P be an aggregate program, (I^c, J^c) be a four-valued interpretation, E be a set of predicates, and P' be the program obtained from P by removing all rules r with $\mathrm{pred}(B(r)^-) \cap E \neq \emptyset$.

We define the *approximate model of P relative* to (I^c, J^c) as

$$AM_E^{(I^c,J^c)}(P) = (I, J)$$

where

$$I = S^{I^c}_{\tau^*(P')}(J^c) \text{ and}$$
$$J = S^{J^c}_{\tau^*(P)}(I^c \cup I)$$

We keep dropping parentheses and simply write $AM_E^{I^c,J^c}(P)$ instead of $AM_E^{(I^c,J^c)}(P)$.

The approximate model amounts to an immediate consequence operator, similar to the relative well-founded operator in Definition 5; it refrains from any iterative applications, as used for defining a well-founded model.

More precisely, the relative ID-stable operator is applied twice to obtain the approximate model. This is similar to Van Gelder's alternating transformation [65]. The certain atoms in I are determined by applying the operator to the ground program obtained after removing all rules whose negative body literals comprise externally defined predicates, while the possible atoms J are computed from the entire program by taking the already computed certain atoms in I into account. In this way, the approximate model may result in fewer unknown atoms than the relative well-founded operator when applied to the least precise interpretation (as an easy point of reference). How well we can approximate the certain atoms with the approximate operator depends on the set of external predicates E. When approximating the model of a program P in a sequence, the set E comprises all negative predicates occurring in P for which possible atoms have not yet been fully computed. This leads to fewer certain atoms obtained from the reduced program, $P' = \{r \in P \mid \text{pred}(B(r)^-) \cap E = \emptyset\}$, stripped of all rules from P that have negative body literals whose predicates occur in E.

The next lemma identifies an essential prerequisite for an approximate model of a non-ground program to be an underapproximation of the well-founded model of the corresponding ground program.

Lemma 40. *Let P be an aggregate program, E be a set of predicates, and (I^c, J^c) be a four-valued interpretation.*

If $\text{pred}(H(P)) \cap \text{pred}(B(P)^-) \subseteq E$ *then* $AM_E^{I^c,J^c}(P) \leq_p WM^{I^c,J^c \cup E^c}(\tau^*(P))$ *where E^c is the set of all ground atoms over predicates in E.*

See proof on page 84.

In general, a grounder cannot calculate on-the-fly a well-founded model. Implementing this task efficiently requires an algorithm storing the grounded program, as, for example, implemented in an ASP solver. With Lemma 38, we see that a stratified component can be evaluated by applying the stable model operator once. In fact, modern grounders are able to calculate this operator on-the-fly. For unstratified components, an approximation of the well-founded model is calculated. This is where we use the approximate model, which might be less precise than the well-founded model but can be computed more easily.

With the condition of Lemma 40 in mind, we define the approximate model for an instantiation sequence. We proceed similar to Definition 7 but treat in (14) all atoms over negative predicates that have not been completely defined as external.

Definition 15. Let $(P_i)_{i \in \mathbb{I}}$ be a (refined) instantiation sequence for P.

Then, the *approximated model* of $(P_i)_{i \in \mathbb{I}}$ is

$$AM((P_i)_{i \in \mathbb{I}}) = \bigsqcup_{i \in \mathbb{I}} (I_i, J_i).$$

where

$$E_i = \mathrm{pred}(B(P_i)^{-}) \cap \mathrm{pred}(\bigcup_{i \leq j} H(P_j)), \tag{14}$$

$$(I_i^c, J_i^c) = \bigsqcup_{j<i} (I_j, J_j), \text{ and} \tag{15}$$

$$(I_i, J_i) = AM_{E_i}^{I_i^c, J_i^c}(P_i). \tag{16}$$

Note that the underlying approximate model removes rules containing negative literals over predicates in E_i when calculating certain atoms. This amounts to assuming all ground instances of atoms over E_i to be possible.[6] Compared to (3), however, this additionally includes recursive predicates in (14). The set E_i is empty for stratified components.

The next result extends Lemma 40 and shows that an approximate model of an instantiation sequence constitutes an underapproximation of the well-founded model of the whole ground program.

Theorem 41. *Let $(P_i)_{i \in \mathbb{I}}$ be an instantiation sequence for aggregate program P and $(P_j)_{j \in \mathbb{J}}$ be a refinement of $(P_i)_{i \in \mathbb{I}}$.*

Then, $AM((P_i)_{i \in \mathbb{I}}) \leq_p AM((P_j)_{j \in \mathbb{J}}) \leq_p WM(\tau^*(P))$.

See proof on page 84.

The finer granularity of refined instantiation sequences leads to more precise models. Intuitively, this is because a refinement of a component may result in a series of approximate models, which yield a more precise result than the approximate model of the entire component because in some cases fewer predicates are considered external in (14).

We remark that all instantiation sequences of a program have the same approximate model. However, this does not carry over to refined instantiation sequences because their evaluation is order dependent.

The two former issues are illustrated in Example 14.

Whenever an aggregate program is stratified, the approximate model of its instantiation sequence is total (and coincides with the well-founded model of the entire ground program).

Theorem 42. *Let $(P_i)_{i \in \mathbb{I}}$ be an instantiation sequence of an aggregate program P such that $E_i = \emptyset$ for each $i \in \mathbb{I}$ as defined in* (14).

Then, $AM((P_i)_{i \in \mathbb{I}})$ is total.

See proof on page 85.

The actual value of approximate models for grounding lies in their underlying series of consecutive interpretations delineating each ground program in a (refined) instantiation sequence. In fact, as outlined after Proposition 35,

[6] To be precise, rules involving aggregates that could in principle derive certain atoms might be removed, too. Here, we are interested in a syntactic criteria that allows us to underapproximate the set of certain atoms.

whenever all interpretations (I_i, J_i) in (16) are finite so are the $\mathcal{R}$-programs $\tau^*_{J_i^c \cup J_i}(P_i)^{(I_i^c, J_i^c) \sqcup (I_i, J_i)}$ obtained from each P_i in the instantiation sequence.

Theorem 43. *Let $(P_i)_{i \in \mathbb{I}}$ be a (refined) instantiation sequence of an aggregate program P, and let (I_i^c, J_i^c) and (I_i, J_i) defined as in (15) and (16).*

Then, $\bigcup_{i \in \mathbb{I}} \tau^*_{J_i^c \cup J_i}(P_i)^{(I_i^c, J_i^c) \sqcup (I_i, J_i)}$ and $\tau^*(P)$ have the same well-founded and stable models.

See proof on page 85.

This union of $\mathcal{R}$-programs is exactly the one obtained by the grounding algorithm proposed in the next section (cf. Theorem 49).

Example 14. The following example shows different ways to split an aggregate program into sequences and gives the well-founded models and approximated models for them. Let P be the following (aggregate) program, extending the one from the introductory section:

$$
\begin{array}{ll}
u(1) & u(2) \\
v(2) & v(3) \\
p(X) \leftarrow \neg q(X) \wedge u(X) & q(X) \leftarrow \neg p(X) \wedge v(X) \\
x \leftarrow \neg p(1) & y \leftarrow \neg q(3)
\end{array}
$$

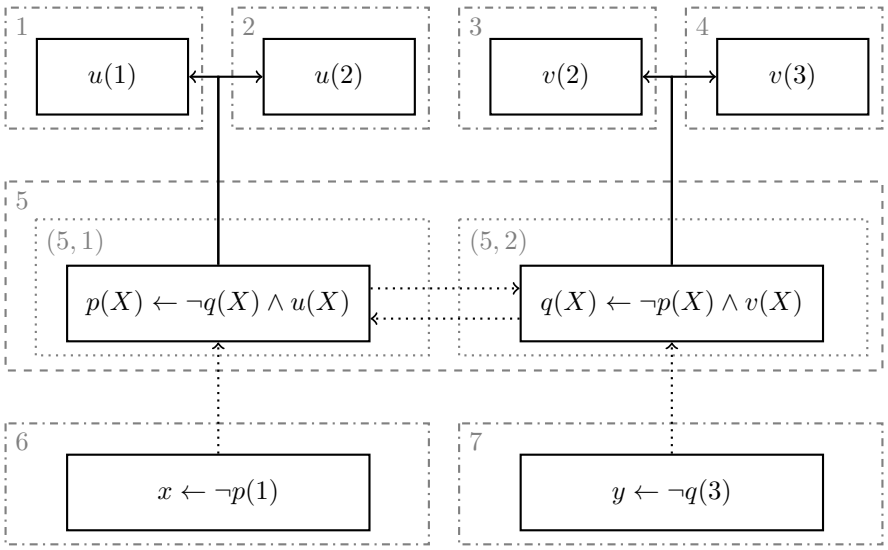

FIGURE 1. Rule dependencies for Example 14.

The (refined) instantiation sequence for program P is given in Figure 1. Rules are depicted in solid boxes. Solid and dotted edges between such boxes depict positive and negative dependencies between the corresponding rules, respectively. A dashed box represents a component in an instantiation sequence and a dotted box a component in a refined instantiation sequence.

If two components coincide, then they are depicted with a dashed/dotted box. The number (or pair) in the corner of a component box indicates the index in the corresponding (refined) instantiation sequence.

For $F = \{u(1), u(2), v(2), v(3)\}$, the well-founded model of $\tau^*(P)$ is

$$WM(\tau^*(P)) = (\{p(1), q(3)\}, \{p(1), p(2), q(2), q(3)\}) \sqcup F.$$

Note that the set F comprises the facts derived from stratified components. For example, we can use Lemma 38 to obtain $I_1 = J_1 = \{u(1)\}$ for component P_1, which also corresponds to the approximate model because the set E_1 in (14) is empty for stratified components.

By Corollary 26, the ground sequence $(\tau(P_i))_{i\in\mathbb{I}}$ has the same well-founded model as $\tau^*(P)$:

$$WM((P_i)_{i\in\mathbb{I}}) = (\{p(1), q(3)\}, \{p(1), p(2), q(2), q(3)\}) \sqcup F.$$

However, the approximated model of the instantiation sequence $(P_i)_{i\in\mathbb{I}}$, defined in Definition 15, is less precise, viz.

$$AM((P_i)_{i\in\mathbb{I}}) = (\emptyset, \{p(1), p(2), q(2), q(3), x, y\}) \sqcup F.$$

This is because we have to use $AM_E^{F,F}(P_5)$ to approximate the well-founded model of component P_5. Here, the set $E = \{a/1, b/1\}$ determined by (14) forces us to unconditionally assume instances of $\neg q(X)$ and $\neg p(X)$ to be true. Thus, we get $(I_5, J_5) = (\emptyset, \{p(1), p(2), q(2), q(3)\})$ for the intermediate interpretation in (16). This is also reflected in Definition 14, which makes us drop all rules containing negative literals over predicates in E when calculating true atoms.

Unlike above, the well-founded model of the refined sequence of ground programs $(\tau(P_{i,j}))_{(i,j)\in\mathbb{J}}$ is

$$WM((\tau(P_{i,j}))_{(i,j)\in\mathbb{J}}) = (\{q(3)\}, \{p(1), p(2), q(2), q(3), x\}) \sqcup F,$$

which is actually less precise than the well-founded model of P. This is because literals over $\neg q(X)$ are unconditionally assumed to be true because their instantiation is not yet available when $P_{5,1}$ is considered. Thus, we get $(I_{5,1}, J_{5,1}) = (\emptyset, \{p(1), p(2)\})$ for the intermediate interpretation in (5). Unlike this, the atom $p(3)$ is false when considering component $P_{5,2}$ and $q(3)$ becomes true. In fact, we get $(I_{5,2}, J_{5,2}) = (\{q(3)\}, \{q(2), q(3)\})$. Observe that (I_5, J_5) from above is less precise than $(I_{5,1}, J_{5,1}) \sqcup (I_{5,2}, J_{5,2})$.

In accord with Theorem 41, we approximate the well-founded model w.r.t. the refined instantiation sequence $(P_{i,j})_{(i,j)\in\mathbb{J}}$ and obtain

$$AM((P_{i,j})_{(i,j)\in\mathbb{J}}) = (\{q(3)\}, \{p(1), p(2), q(2), q(3), x\}) \sqcup F,$$

which, for this example, is equivalent to the well-founded model of the corresponding ground refined instantiation sequence and more precise than the approximate model of the instantiation sequence.

Remark 1. The reason why we use the refined grounding is that we cannot expect a grounding algorithm to calculate the well-founded model for a

component without further processing. But at least some consequences should be considered. *Gringo* is designed to ground on-the-fly without storing any rules, so it cannot be expected to compute all possible consequences but it should at least take all consequences from preceding interpretations into account. With the help of a solver, we could calculate the exact well-founded model of a component after it has been grounded.

Example 15. The dependency graph of the company controls encoding is given in Figure 2 and follows the conventions in Example 14. Because the encoding only uses positive literals and monotone aggregates, grounding sequences cannot be refined further. Since the program is positive, we can apply Theorem 42. Thus, the approximate model of the grounding sequence is total and corresponds to the well-founded model of the program. We use the same abbreviations for predicates as in Figure 2. The well-founded model is $(F \cup I, F \cup I)$ where

$$\begin{aligned} F = \{&c(c_1), c(c_2), c(c_3), c(c_4), \\ &o(c_1, c_2, 60), o(c_1, c_3, 20), o(c_2, c_3, 35), o(c_3, c_4, 51)\} \text{ and} \\ I = \{&s(c_1, c_2), s(c_3, c_4), s(c_1, c_3), s(c_1, c_4)\}. \end{aligned}$$

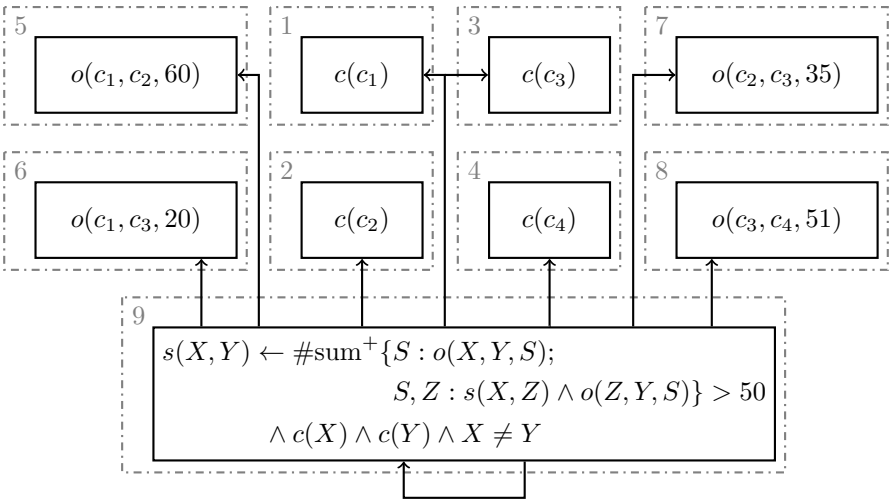

FIGURE 2. Rule dependencies for the company controls encoding and instance in Tables 1 and 2 where $c = \textit{company}$, $o = \textit{owns}$, and $s = \textit{controls}$.

6. ALGORITHMS

This section lays out the basic algorithms for grounding rules, components, and entire programs and characterizes their output in terms of the semantic concepts developed in the previous sections. Of particular interest is the

treatment of aggregates, which are decomposed into dedicated normal rules before grounding, and re-assembled afterwards. This allows us to ground rules with aggregates by means of grounding algorithms for normal rules.

In the following, we refer to terms, atoms, comparisons, literals, aggregate elements, aggregates, or rules as *expressions*. As in the preceding sections, all expressions, interpretations, and concepts introduced below operate on the same (implicit) signature Σ unless mentioned otherwise.

A *substitution* is a mapping from the variables in Σ to terms over Σ. We use ι to denote the identity substitution mapping each variable to itself. A *ground substitution* maps all variables to ground terms or themselves. The result of *applying* a substitution σ to an expression e, written $e\sigma$, is the expression obtained by replacing each variable v in e by $\sigma(v)$. This directly extends to sets E of expressions, that is, $E\sigma = \{e\sigma \mid e \in E\}$.

The *composition* of substitutions σ and θ is the substitution $\sigma \circ \theta$ where $(\sigma \circ \theta)(v) = \theta(\sigma(v))$ for each variable v.

A substitution σ is a *unifier* of a set E of expressions if $e_1\sigma = e_2\sigma$ for all $e_1, e_2 \in E$. In what follows, we are interested in one-sided unification, also called *matching*. A substitution σ matches a non-ground expression e to a ground expression g, if $e\sigma = g$ and σ maps all variables not occurring in e to themselves. We call such a substitution the *matcher* of e to g. Note that a matcher is a unique ground substitution unifying e and g, if it exists. This motivates the following definition.

For a (non-ground) expression e and a ground expression g, we define:

$$\texttt{match}(e, g) = \begin{cases} \{\sigma\} & \text{if there is a matcher } \sigma \text{ from } e \text{ to } g \\ \emptyset & \text{otherwise} \end{cases}$$

When grounding rules, we look for matches of non-ground body literals in the Herbrand base accumulated so far. The latter is captured by a four-valued interpretation to distinguish certain atoms among the possible ones. This is made precise in the next definition.

Definition 16. Let σ be a substitution, l be a literal or comparison, and (I, J) be a four-valued interpretation.

We define the set of *matches* for l in (I, J) w.r.t. σ

for an atom $l = a$ as

$$\texttt{Matches}_a^{I,J}(\sigma) = \{\sigma \circ \sigma' \mid a' \in J, \sigma' \in \texttt{match}(a\sigma, a')\},$$

for a ground literal $l = \neg a$ as

$$\texttt{Matches}_{\neg a}^{I,J}(\sigma) = \{\sigma \mid a\sigma \notin I\}, \text{ and}$$

for a ground comparison $l = t_1 \prec t_2$ as in (6) as

$$\texttt{Matches}_{t_1 \prec t_2}^{I,J}(\sigma) = \{\sigma \mid \prec \text{ holds between } t_1\sigma \text{ and } t_2\sigma\}.$$

In this way, positive body literals yield a (possibly empty) set of substitutions, refining the one at hand, while negative and comparison literals are only considered when ground and then act as a test on the given substitution.

```
1  function GroundRule^{I,J}_{r,f,J'}(σ, L)
2      if L ≠ ∅ then                                          // match next
3          (G, l) ← (∅, Select_σ(L));
4          foreach σ' ∈ Matches^{I,J}_l(σ) do
5              G ← G ∪ GroundRule^{I,J}_{r,f,J'}(σ', L \ {l});
6          return G;
7      else if f = t or B(rσ)^+ ⊈ J' then                  // rule instance
8          return {rσ};
9      else                                                 // rule seen
10         return ∅;
```

Algorithm 1: Grounding Rules

Our function for rule instantiation is given in Algorithm 1. It takes a substitution σ and a set L of literals and yields a set of ground instances of a safe normal rule r, passed as a parameter; it is called from Algorithm 2 with the identity substitution and the body literals $B(r)$ of r. The other parameters consist of a four-valued interpretation (I, J) comprising the current Herbrand base along with its certain atoms, a two-valued interpretation J' reflecting the previous value of J, and a Boolean flag f used to avoid duplicate ground rules in consecutive calls to Algorithm 1. The idea is to extend the current substitution in Lines 4 to 5 until we obtain a ground substitution σ that induces a ground instance $r\sigma$ of rule r. To this end, $\texttt{Select}_\sigma(L)$ picks for each call some literal $l \in L$ such that $l \in L^+$ or $l\sigma$ is ground. That is, it yields either a positive body literal or a ground negative or ground comparison literal, as needed for computing $\texttt{Matches}^{I,J}_l(\sigma)$. Whenever an applications of `Matches` for the selected literal in $B(r)$ results in a non-empty set of substitutions, the function is called recursively for each such substitution. The recursion terminates if at least one match is found for each body literal and an instance $r\sigma$ of r is obtained in Line 8. The set of all such ground instances is returned in Line 6. (Note that we refrain from applying any simplifications to the ground rules and rather leave them intact to obtain more direct formal characterizations of the results of our grounding algorithms.) The test $B(r\sigma)^+ \not\subseteq J'$ in Line 7 makes sure that no ground rules are generated that were already obtained by previous invocations of Algorithm 1. This is relevant for recursive rules and reflects the approach of semi-naive database evaluation [1].

We can characterize the result of Algorithm 1 as follows.

Proposition 44. *Let r be a safe normal rule, (I, J) be a finite four-valued interpretation, $f \in \{\mathbf{t}, \mathbf{f}\}$, and J' be a finite two-valued interpretation.*

Then, a call to $\texttt{GroundRule}^{I,J}_{r,f,J'}(\iota, B(r))$ *returns the finite set of instances* g *of* r *satisfying*

$$J \models \tau(B(g))^{\wedge}_{I} \text{ and } (f = \mathbf{t} \text{ or } B(g)^{+} \not\subseteq J'). \tag{17}$$

See proof on page 86.

In terms of the program simplification in Definition 1, the first condition amounts to checking whether $H(g) \leftarrow \tau(B(g))^{\wedge}$ is in $\tau(P)^{I,J}$, which is the simplification of the (ground) $\mathcal{R}$-program $\tau(P)$ preserving all stable models between I and J. The two last conditions are meant to avoid duplicates from a previous invocation. Since r is a normal rule, translation τ is sufficient.

For characterizing the result of Algorithm 1 in terms of aggregate programs, we need the following definition.

Definition 17. Let P be an aggregate program and (I, J) be a four-valued interpretation.

We define $\mathrm{Inst}^{I,J}(P)$ as the set of all instances g of rules in P satisfying $J \models \tau^{*}(B(g))^{\wedge}_{I}$.

Similar to above, an instance g belongs to $\mathrm{Inst}^{I,J}(P)$ iff $H(g) \leftarrow \tau^{*}(B(g))^{\wedge} \in \tau^{*}(r)^{I,J}$. Note that the members of $\mathrm{Inst}^{I,J}(P)$ are not necessarily ground, since non-global variables may remain within aggregates; though they are ground for normal rules.

Algorithm 1 is called consecutively within a loop in Algorithm 2. The purpose of the Boolean flag f is to ensure that initially all rules are grounded. In subsequent iterations, duplicates are omitted by setting the flag to false and filtering rules whose positive bodies are a subset of the atoms J' used in previous iterations.

In fact, when grounding from scratch with $J' = \emptyset$ and $f = \mathbf{t}$, the right hand side of (17) just like the test in Line 7 are satisfied:

Corollary 45. *Let* r *be a safe normal rule and* (I, J) *be a finite four-valued interpretation.*

Then, $\mathrm{Inst}^{I,J}(\{r\}) = \texttt{GroundRule}^{I,J}_{r,\mathbf{t},\emptyset}(\iota, B(r))$.

See proof on page 88.

This relation can be seen as the base case of an iterative construction.

In view of this, the next result shows how subsequent iterations extend previous ones.

Lemma 46. *Let* r *be a safe normal rule,* (I, J) *be a finite four-valued interpretation, and* $J' \subseteq J$ *be a two-valued interpretation.*

Then, we have

$$\mathrm{Inst}^{I,J}(\{r\}) = \mathrm{Inst}^{I,J'}(\{r\}) \cup \texttt{GroundRule}^{I,J}_{r,\mathbf{f},J'}(\iota, B(r)).$$

See proof on page 89.

Now, let us turn to the treatment of aggregates. To this end, we define the following translation of aggregate programs to normal programs.

Definition 18. Let P be a safe aggregate program over signature Σ.

Let Σ' be the signature obtained by extending Σ with fresh predicates

$$\alpha_{a,r}/n, \text{ and} \tag{18}$$

$$\epsilon_{a,r}/n \tag{19}$$

for each aggregate a occurring in a rule $r \in P$ where n is the number of global variables in a, and fresh predicates

$$\eta_{e,a,r}/(m+n) \tag{20}$$

for each aggregate element e occurring in aggregate a in rule r where m is the size of the tuple $H(e)$.

We define P^α, P^ϵ, and P^η as normal programs over Σ' as follows.

- Program P^α is obtained from P by replacing each aggregate occurrence a in P with

$$\alpha_{a,r}(X_1, \ldots, X_n) \tag{21}$$

  where $\alpha_{a,r}/n$ is defined as in (18) and $X_1, \ldots, X_n$ are the global variables in a.
- Program P^ϵ consists of rules

$$\epsilon_{a,r}(X_1, \ldots, X_n) \leftarrow t \prec b \wedge b_1 \wedge \cdots \wedge b_l \tag{22}$$

  for each predicate $\epsilon_{a,r}/n$ as in (19) where $X_1, \ldots, X_n$ are the global variables in a, a is an aggregate of form $f\{E\} \prec b$ occurring in r, $t = f(\emptyset)$ is the value of the aggregate function applied to the empty set, and $b_1, \ldots, b_l$ are the body literals of r excluding aggregates.
- Program P^η consists of rules

$$\eta_{e,a,r}(t_1, \ldots, t_m, X_1, \ldots, X_n) \leftarrow c_1 \wedge \cdots \wedge c_k \wedge b_1 \wedge \cdots \wedge b_l \tag{23}$$

  for each predicate $\eta_{e,a,r}/m+n$ as in (20) where $t_1, \ldots, t_m = H(e)$, $X_1, \ldots, X_n$ are the global variables in a, $c_1, \ldots, c_k = B(e)$, and $b_1, \ldots, b_l$ are the body literals of r excluding aggregates.

Summarizing the above, we translate an aggregate program P over Σ into a normal program P^α along with auxiliary normal rules in P^ϵ and P^η, all over a signature extending Σ by the special-purpose predicates in (18) to (20). In fact, there is a one-to-one correspondence between the rules in P and P^α, so that we get $P = P^\alpha$ and $P^\epsilon = P^\eta = \emptyset$ whenever P is already normal.

Example 16. We illustrate the translation of aggregate programs on the company controls example in Table 1. We rewrite the rule $r =$

$$\mathit{controls}(X,Y) \leftarrow \overbrace{\#\mathrm{sum}^+\{\underbrace{S : \mathit{owns}(X,Y,S)}_{e_1};\ \underbrace{S,Z : \mathit{controls}(X,Z) \wedge \mathit{owns}(Z,Y,S)}_{e_2}\} > 50}^{a}$$
$$\wedge\ \mathit{company}(X) \wedge \mathit{company}(Y) \wedge X \neq Y$$

containing an aggregate a, into rules r_1 to r_4:

$$(r_1) \qquad \mathit{controls}(X,Y) \leftarrow \alpha_{a,r}(X,Y) \wedge \mathit{company}(X) \wedge \mathit{company}(Y) \wedge X \neq Y,$$

$$(r_2) \qquad \epsilon_{a,r}(X,Y) \leftarrow 0 > 50 \wedge \mathit{company}(X) \wedge \mathit{company}(Y) \wedge X \neq Y,$$

$$(r_3) \qquad \eta_{e_1,a,r}(S,X,Y) \leftarrow \mathit{owns}(X,Y,S) \wedge \mathit{company}(X) \wedge \mathit{company}(Y) \wedge X \neq Y, \text{ and}$$

$$(r_4) \qquad \eta_{e_2,a,r}(S,Z,X,Y) \leftarrow \mathit{controls}(X,Z) \wedge \mathit{owns}(Z,Y,S) \wedge \mathit{company}(X) \wedge \mathit{company}(Y) \wedge X \neq Y.$$

We have $P^\alpha = \{r_1\}$, $P^\epsilon = \{r_2\}$, and $P^\eta = \{r_3, r_4\}$.

This example nicely illustrates how possible instantiations of aggregate elements are gathered via the rules in P^η. Similarly, the rules in P^ϵ collect instantiations warranting that the result of applying aggregate functions to the empty set is in accord with the respective bound. In both cases, the relevant variable bindings are captured by the special head atoms of the rules. In turn, groups of corresponding instances of aggregate elements are used in Definition 21 to sanction the derivation of ground atoms of form (21). These atoms are ultimately replaced in P^α with the original aggregate contents.

We next define two functions gathering information from instances of rules in P^ϵ and P^η. In particular, we make precise how groups of aggregate element instances are obtained from ground rules in P^η.

Definition 19. Let P be an aggregate program, and G^ϵ and G^η be subsets of ground instances of rules in P^ϵ and P^η, respectively. Furthermore, let a be an aggregate occurring in some rule $r \in P$ and σ be a substitution mapping the global variables in a to ground terms.

We define

$$\epsilon_{r,a}(G^\epsilon, \sigma) = \bigcup_{g \in G^\epsilon} \mathtt{match}(r_a\sigma, g)$$

where r_a is a rule of form (22) for aggregate occurrence a, and

$$\eta_{r,a}(G^\eta, \sigma) = \{e\sigma\theta \mid g \in G^\eta, e \in E, \theta \in \mathtt{match}(r_e\sigma, g)\}$$

where E are the aggregate elements of a and r_e is a rule of form (23) for aggregate element occurrence e in a.

Given that σ maps the global variables in a to ground terms, $r_a\sigma$ is ground whereas $r_e\sigma$ may still contain local variables from a. The set $\epsilon_{r,a}(G^\epsilon, \sigma)$ has an indicative nature: For an aggregate $a\sigma$, it contains the identity substitution when the result of applying its aggregate function to the empty set is in accord with its bound, and it is empty otherwise. The construction of $\eta_{r,a}(G^\eta, \sigma)$ goes one step further and reconstitutes all ground aggregate elements of $a\sigma$ from variable bindings obtained by rules in G^η. Both functions play a central role below in defining the function `Propagate` for deriving ground aggregate placeholders of form (21) from ground rules in G^ϵ and G^η.

Example 17. We show how to extract aggregate elements from ground instances of rules (r_3) and (r_4) in Example 16.

Let G^ϵ be empty and G^η be the program consisting of the following rules:

$$\begin{aligned}
\eta_{e_1,a,r}(60, c_1, c_2) &\leftarrow \mathit{owns}(c_1, c_2, 60) \\
&\quad \wedge \mathit{company}(c_1) \wedge \mathit{company}(c_2) \wedge c_1 \neq c_2, \\
\eta_{e_1,a,r}(20, c_1, c_3) &\leftarrow \mathit{owns}(c_1, c_3, 20) \\
&\quad \wedge \mathit{company}(c_1) \wedge \mathit{company}(c_3) \wedge c_1 \neq c_3, \\
\eta_{e_1,a,r}(35, c_2, c_3) &\leftarrow \mathit{owns}(c_2, c_3, 35) \\
&\quad \wedge \mathit{company}(c_2) \wedge \mathit{company}(c_3) \wedge c_2 \neq c_3, \\
\eta_{e_1,a,r}(51, c_3, c_4) &\leftarrow \mathit{owns}(c_3, c_4, 51) \\
&\quad \wedge \mathit{company}(c_3) \wedge \mathit{company}(c_4) \wedge c_3 \neq c_4, \text{ and} \\
\eta_{e_2,a,r}(35, c_2, c_1, c_3) &\leftarrow \mathit{controls}(c_1, c_2) \wedge \mathit{owns}(c_2, c_3, 35) \\
&\quad \wedge \mathit{company}(c_1) \wedge \mathit{company}(c_3) \wedge c_1 \neq c_3.
\end{aligned}$$

Clearly, we have $\epsilon_{r,a}(G^\epsilon, \sigma) = \emptyset$ for any substitution σ because $G^\epsilon = \emptyset$. This means that aggregate a can only be satisfied if at least one of its elements is satisfiable. In fact, we obtain non-empty sets $\eta_{r,a}(G^\eta, \sigma)$ of ground aggregate elements for four substitutions σ:

$$\begin{aligned}
\eta_{r,a}(G^\eta, \sigma_1) &= \{60 : \mathit{owns}(c_1, c_2, 60)\} && \text{for } \sigma_1 : X \mapsto c_1, Y \mapsto c_2, \\
\eta_{r,a}(G^\eta, \sigma_2) &= \{51 : \mathit{owns}(c_3, c_4, 51)\} && \text{for } \sigma_2 : X \mapsto c_3, Y \mapsto c_4, \\
\eta_{r,a}(G^\eta, \sigma_3) &= \{35, c_2 : \mathit{controls}(c_1, c_2) \wedge \mathit{owns}(c_2, c_3, 35); \\
&\qquad 20 : \mathit{owns}(c_1, c_3, 20)\} && \text{for } \sigma_3 : X \mapsto c_1, Y \mapsto c_3, \text{ and} \\
\eta_{r,a}(G^\eta, \sigma_4) &= \{35 : \mathit{owns}(c_2, c_3, 35)\} && \text{for } \sigma_4 : X \mapsto c_2, Y \mapsto c_3.
\end{aligned}$$

For capturing the result of grounding aggregates relative to groups of aggregate elements gathered via P^η, we restrict their original translation to

subsets of their ground elements. That is, while $\tau^*(a)$ and $\tau^*_a(\cdot)$ draw in Definition 10 on all instances of aggregate elements in a, their counterparts $\tau^*_G(a)$ and $\tau^*_{a,G}(\cdot)$ are restricted to a subset of such aggregate element instances:[7]

Definition 20. Let a be a closed aggregate and of form (8) and let E be the set of its aggregate elements. Let $G \subseteq \text{Inst}(E)$ be a set of aggregate element instances.

We define the translation $\tau^*_G(a)$ of a w.r.t. G as follows:

$$\tau^*_G(a) = \{\tau(D)^\wedge \rightarrow \tau^*_{a,G}(D)^\vee \mid D \subseteq G, D \not\models a\}^\wedge$$

where

$$\tau^*_{a,G}(D) = \{\tau(C)^\wedge \mid C \subseteq G \setminus D, C \cup D \triangleright a\}.$$

As before, this translation maps aggregates, possibly including non-global variables, to a conjunction of (ground) $\mathcal{R}$-rules. The resulting $\mathcal{R}$-formula is used below in the definition of functions `Assemble` and `Propagate`.

Example 18. We consider the four substitutions σ_1 to σ_4 together with the sets $G_1 = \eta_{r,a}(G^\eta, \sigma_1)$ to $G_4 = \eta_{r,a}(G^\eta, \sigma_4)$ from Example 17 for aggregate a.

Following the discussion after Proposition 34, we get the formulas:

$\tau^*_{G_1}(a\sigma_1) = owns(c_1, c_2, 60)$,
$\tau^*_{G_2}(a\sigma_2) = owns(c_3, c_4, 51)$,
$\tau^*_{G_3}(a\sigma_3) = controls(c_1, c_2) \wedge owns(c_2, c_3, 35) \wedge owns(c_1, c_3, 20)$, and
$\tau^*_{G_4}(a\sigma_4) = \bot$.

Observe that the first three formulas capture the first three aggregates in Table 3,

The function `Propagate` yields a set of ground atoms of form (21) that are used in Algorithm 2 to ground rules having such placeholders among their body literals. Each such special atom is supported by a group of ground instances of its aggregate elements.

Definition 21. Let P be an aggregate program, (I, J) be a four-valued interpretation, and G^ϵ and G^η be subsets of ground instances of rules in P^ϵ and P^η, respectively.

We define $\texttt{Propagate}^{I,J}_P(G^\epsilon, G^\eta)$ as the set of all atoms of form $\alpha\sigma$ such that $\epsilon_{r,a}(G^\epsilon, \sigma) \cup G \neq \emptyset$ and $J \models \tau^*_G(a\sigma)_I$ with $G = \eta_{r,a}(G^\eta, \sigma)$ where α is an atom of form (21) for aggregate a in rule r and σ is a ground substitution for r mapping all global variables in a to ground terms.

An atom $\alpha\sigma$ is only considered if σ warrants ground rules in G^ϵ or G^η, signaling that the application of α to the empty set is feasible when applying σ or that there is a non-empty set of ground aggregate elements of α obtained

[7]Note that the restriction to sets of ground aggregate elements is similar to the one to two-valued interpretations in Definition 11.

after applying σ, respectively. If this is the case, it is checked whether the set G of aggregate element instances warrants that $\tau^*_G(a\sigma)$ admits stable models between I and J.

Example 19. We show how to propagate aggregates using the sets G_1 to G_4 and their associated formulas from Example 18. Suppose that $I = J = F \cup \{\mathit{controls}(c_1, c_2)\}$ using F from Example 15.

Observe that $J \models \tau^*_{G_1}(a\sigma_1)_I$, $J \models \tau^*_{G_2}(a\sigma_2)_I$, $J \models \tau^*_{G_3}(a\sigma_3)_I$, and $J \not\models \tau^*_{G_4}(a\sigma_4)_I$. Thus, we get $\texttt{Propagate}^{I,J}_P(G^\epsilon, G^\eta) = \{\alpha_{a,r}(c_1, c_2), \alpha_{a,r}(c_1, c_3), \alpha_{a,r}(c_3, c_4)\}$.

The function `Assemble` yields an $\mathcal{R}$-program in which aggregate placeholder atoms of form (21) have been replaced by their corresponding $\mathcal{R}$-formula.

Definition 22. Let P be an aggregate program, and G^α and G^η be subsets of ground instances of rules in P^α and P^η, respectively.

We define $\texttt{Assemble}(G^\alpha, G^\eta)$ as the $\mathcal{R}$-program obtained from G^α by replacing

- all comparisons by $\top$ and
- all atoms of form $\alpha\sigma$ by the corresponding formulas $\tau^*_G(a\sigma)$ with $G = \eta_{r,a}(G^\eta, \sigma)$ where α is an atom of form (21) for aggregate a in rule r and σ is a ground substitution for r mapping all global variables in a to ground terms.

Example 20. We show how to assemble aggregates using the sets G_1 to G_3 for aggregate atoms that have been propagated in Example 19. Therefore, let G^α be the program consisting of the following rules:

$$\mathit{controls}(c_1, c_2) \leftarrow \alpha_{a,r}(c_1, c_2) \wedge \mathit{company}(c_1) \wedge \mathit{company}(c_2) \wedge c_1 \neq c_2,$$
$$\mathit{controls}(c_3, c_4) \leftarrow \alpha_{a,r}(c_3, c_4) \wedge \mathit{company}(c_3) \wedge \mathit{company}(c_4) \wedge c_3 \neq c_4, \text{ and}$$
$$\mathit{controls}(c_1, c_3) \leftarrow \alpha_{a,r}(c_1, c_3) \wedge \mathit{company}(c_1) \wedge \mathit{company}(c_3) \wedge c_1 \neq c_3.$$

Then, program $\texttt{Assemble}(G^\alpha, G^\eta)$ consists of the following rules:

$$\mathit{controls}(c_1, c_2) \leftarrow \tau^*_{G_1}(a\sigma_1) \wedge \mathit{company}(c_1) \wedge \mathit{company}(c_2) \wedge \top,$$
$$\mathit{controls}(c_3, c_4) \leftarrow \tau^*_{G_2}(a\sigma_2) \wedge \mathit{company}(c_3) \wedge \mathit{company}(c_4) \wedge \top, \text{ and}$$
$$\mathit{controls}(c_1, c_3) \leftarrow \tau^*_{G_3}(a\sigma_3) \wedge \mathit{company}(c_1) \wedge \mathit{company}(c_3) \wedge \top.$$

The next result shows how a (non-ground) aggregate program P is transformed into a (ground) $\mathcal{R}$-program $\tau^*_J(P)^{I,J}$ in the context of certain and possible atoms (I, J) via the interplay of grounding P^ϵ and P^η, deriving aggregate placeholders from their ground instances G^ϵ and G^η, and finally replacing them in G^α by the original aggregates' contents.

Proposition 47. *Let P be an aggregate program, (I, J) be a finite four-valued interpretation, $G^\epsilon = \mathrm{Inst}^{I,J}(P^\epsilon)$, $G^\eta = \mathrm{Inst}^{I,J}(P^\eta)$, $J^\alpha = \texttt{Propagate}^{I,J}_P(G^\epsilon, G^\eta)$, and $G^\alpha = \mathrm{Inst}^{I,J\cup J^\alpha}(P^\alpha)$.*

Then,

(a) $\texttt{Assemble}(G^\alpha, G^\eta) = \tau^*_J(P)^{I,J}$ *and*
(b) $H(G^\alpha) = T_{\tau^*(P)_I}(J)$.

See proof on page 89.

Property (b) highlights the relation of the possible atoms contributed by G^α to a corresponding application of the immediate consequence operator. In fact, this is the first of three such relationships between grounding algorithms and consequence operators.

```
1  function GroundComponent(P, I, J)
2      (G^α, G^ϵ, G^η, f, J^α, J^α', J') ← (∅, ∅, ∅, t, ∅, ∅, ∅);
3      repeat
           // ground aggregate elements
4          G^ϵ ← G^ϵ ∪ ⋃_{r∈P^ϵ} GroundRule^{I,J}_{r,f,J'}(ι, B(r));
5          G^η ← G^η ∪ ⋃_{r∈P^η} GroundRule^{I,J}_{r,f,J'}(ι, B(r));
           // propagate aggregates
6          J^α ← Propagate^{I,J}_P(G^ϵ, G^η);
           // ground remaining rules
7          G^α ← G^α ∪ ⋃_{r∈P^α} GroundRule^{I,J∪J^α}_{r,f,J'∪J^α'}(ι, B(r));
8          (f, J^α', J', J) ← (f, J^α, J, J ∪ H(G^α));
9      until J' = J;
10     return Assemble(G^α, G^η);
```

Algorithm 2: Grounding Components

Let us now turn to grounding components of instantiation sequences in Algorithm 2. The function `GroundComponent` takes an aggregate program P along with two sets I and J of ground atoms. Intuitively, P is a component in a (refined) instantiation sequence and I and J form a four-valued interpretation (I, J) comprising the certain and possible atoms gathered while grounding previous components (although their roles get reversed in Algorithm 3). After variable initialization, `GroundComponent` loops over consecutive rule instantiations in P^α, P^ϵ, and P^η until no more possible atoms are obtained. In this case, it returns in Line 10 the $\mathcal{R}$-program obtained from G^α by replacing all ground aggregate placeholders of form (21) with the $\mathcal{R}$-formula corresponding to the respective ground aggregate. The body of the loop can be divided into two parts: Lines 4 to 6 deal with aggregates and Lines 7 and 8 care about grounding the actual program. In more detail, Lines 4 and 5 instantiate programs P^ϵ and P^η, whose ground instances, G^ϵ and G^η, are then used in Line 6 to derive ground instances of aggregate placeholders of form (21). The grounded placeholders are then added via variable J^α to the possible atoms J when grounding the actual program P^α in Line 7, where J' and $J^{\alpha\prime}$ hold the previous value of J and J^α, respectively. For the next iteration, J is augmented in Line 8 with all rule heads in G^α and the flag f is set to false. Recall that the purpose of f is to ensure

that initially all rules are grounded. In subsequent iterations, duplicates are omitted by setting the flag to false and filtering rules whose positive bodies are a subset of the atoms $J' \cup J^{\alpha'}$ used in previous iterations.

While the inner workings of Algorithm 2 follow the blueprint given by Proposition 47. its outer functionality boils down to applying the ID-stable operator of the corresponding ground program in the context of the certain and possible atoms gathered so far.

Proposition 48. *Let P be an aggregate program, (I^c, J^c) be a finite four-valued interpretation, and $J = S^{J^c}_{\tau^*(P)}(I^c)$.*

Then,

(a) $\texttt{GroundComponent}(P, I^c, J^c)$ *terminates iff J is finite.*

If J is finite, then

(b) $\texttt{GroundComponent}(P, I^c, J^c) = \tau^*_{J^c \cup J}(P)^{I^c, J^c \cup J}$ *and*

(c) $H(\texttt{GroundComponent}(P, I^c, J^c)) = J$.

See proof on page 90.

```
1 function Ground(P)
2     let (P_i)_{i∈𝕀} be a refined instantiation sequence for P;
3     (F, G) ← (∅, ∅);
4     foreach i ∈ 𝕀 do
5         let P'_i be the program obtained from P_i as in Definition 14;
6         F ← F ∪ GroundComponent(P'_i, H(G), H(F));
7         G ← G ∪ GroundComponent(P_i, H(F), H(G));
8     return G;
```

Algorithm 3: Grounding Programs

Finally, Algorithm 3 grounds an aggregate program by iterating over the components of one of its refined instantiation sequences. Just as Algorithm 2 reflects the application of a stable operator, function `Ground` follows the definition of an approximate model when grounding a component (cf. Definition 14). At first, facts are computed in Line 6 by using the program stripped from rules being involved in a negative cycle overlapping with the present or subsequent components. The obtained head atoms are then used in Line 7 as certain context atoms when computing the ground version of the component at hand. The possible atoms are provided by the head atoms of the ground program built so far, and with roles reversed in Line 6. Accordingly, the whole iteration aligns with the approximated model of the chosen refined instantiation sequence (cf. Definition 15), as made precise next.

Theorem 49. *Let P be an aggregate program, $(P_i)_{i \in \mathbb{I}}$ be a refined instantiation sequence for P, and E_i, (I^c_i, J^c_i), and (I_i, J_i) be defined as in* (14) *to* (16).

If $(P_i)_{i \in \mathbb{I}}$ is selected by Algorithm 3 in Line 2, then we have that

(a) *the call* $\texttt{Ground}(P)$ *terminates iff* $AM((P_i)_{i\in\mathbb{I}})$ *is finite, and*
(b) *if* $AM((P_i)_{i\in\mathbb{I}})$ *is finite, then* $\texttt{Ground}(P) = \bigcup_{i\in\mathbb{I}} \tau^*_{J_i^c\cup J_i}(P_i)^{(I_i^c,J_i^c)\sqcup(I_i,J_i)}$.

See proof on page 91.

As already indicated by Theorem 43, grounding is governed by the series of consecutive approximate models (I_i, J_i) in (16) delineating the stable models of each ground program in a (refined) instantiation sequence. Whenever each of them is finite, we also obtain a finite grounding of the original program. Note that the entire ground program is composed of the ground programs of each component in the chosen instantiation sequence. Hence, different sequences may result in different overall ground programs.

Most importantly, our grounding machinery guarantees that an obtained finite ground program has the same stable models as the original non-ground program.

Corollary 50. *Let P be an aggregate program.*

If $\texttt{Ground}(P)$ *terminates then P and* $\texttt{Ground}(P)$ *have the same well-founded and stable models.*

See proof on page 91.

$P'_{5,1} = \emptyset$

$u(X)$	$\neg q(X)$	$p(X)$	1
$u(1)$	$\longrightarrow \neg q(1)$	$\longrightarrow p(1)$	1.1
$u(2)$	$\longrightarrow \neg q(2)$	$\longrightarrow p(2)$	

(A) $I_{5,1} = I_4 \cup H(\texttt{GC}(P'_{5,1}, J_4, I_4))$

$v(X)$	$\neg p(X)$	$q(X)$	1
$v(2)$	$\longrightarrow \times$		1.1
$v(3)$	$\longrightarrow \neg p(3)$	$\longrightarrow q(3)$	

(B) $J_{5,1} = J_4 \cup H(\texttt{GC}(P_{5,1}, I_{5,1}, J_4))$

$v(X)$	$\neg p(X)$	$q(X)$	1
$v(2)$	$\longrightarrow \neg p(2)$	$\longrightarrow q(2)$	1.1
$v(3)$	$\longrightarrow \neg p(3)$	$\longrightarrow q(3)$	

(C) $I_{5,2} = I_{5,1} \cup H(\texttt{GC}(P'_{5,2}, J_{5,1}, I_{5,1}))$

$\neg p(1)$	x	1
$\times$		1.1

(D) $J_{5,2} = J_{5,1} \cup H(\texttt{GC}(P_{5,2}, I_{5,2}, J_{5,1}))$

$\neg p(1)$	x	1
$\neg p(1)$	$\longrightarrow x$	1.1

(E) $I_6 = I_{5,2} \cup H(\texttt{GC}(P'_6, J_{5,2}, I_{5,2}))$

$\neg q(3)$	y	1
$\times$		1.1

(F) $J_6 = J_{5,2} \cup H(\texttt{GC}(P_6, I_6, J_{5,2}))$

$\neg q(3)$	y	1
$\times$		1.1

(G) $I_7 = I_6 \cup H(\texttt{GC}(P'_7, J_6, I_6))$

(H) $J_7 = J_6 \cup H(\texttt{GC}(P_7, I_7, J_6))$

TABLE 4. Grounding of components $P_{5,1}$, $P_{5,2}$, P_6, and P_7 from Example 14 where $\texttt{GC} = \texttt{GroundComponent}$.

Example 21. The execution of the grounding algorithms on Example 14 is illustrated in Table 4. Each table depicts a call to `GroundComponent` where the header above the double line contains the (literals of the) rules to be grounded and the rows below trace how nested calls to `GroundRule`

proceed. The rules in the header contain the body literals in the order as they are selected by `GroundRule` with the rule head as the last literal. Calls to `GroundRule` are depicted with vertical lines and horizontal arrows. A vertical line represents the iteration of the loop in Lines 4 to 5. A horizontal arrow represents a recursive call to `GroundRule` in Line 5. Each row in the table not marked with $\times$ corresponds to a ground instance as returned by `GroundRule`. Furthermore, because all components are stratified, we only show the first iteration of the loop in Lines 3 to 9 of Algorithm 2 as the second iteration does not produce any new ground instances.

Grounding components P_1 to P_4 results in the theories $F = G = \{u(1) \leftarrow \top, u(2) \leftarrow \top, v(2) \leftarrow \top, v(3) \leftarrow \top\}$. Since grounding is driven by the sets of true and possible atoms, we focus on the interpretations I_i and J_i where i is a component index in the refined instantiation sequence. We start tracing the grounding starting with $I_4 = J_4 = \{u(1), u(2), v(2), v(3)\}$.

The grounding of $P_{5,1}$ as depicted in Tables 4a and 4b. We have $E = \{b/1\}$ because predicate $b/1$ is used in the head of the rule in $P_{5,2}$. Thus, $P'_{5,1} = \emptyset$ and `GroundComponent`$(\emptyset, J_4, I_4)$ in Line 6 returns the empty set and we get $I_{5,1} = I_4$. In the next line, the algorithm calls `GroundComponent`$(P_{5,1}, I_{5,1}, J_4)$ and we get $J_{5,1} = \{p(1), p(2)\}$ Note that at this point, it is not known that $q(1)$ is not derivable and so the algorithm does not derive $p(1)$ as a fact.

The grounding of $P_{5,2}$ is given in Tables 4c and 4d. This time, we have $E = \emptyset$ and $P_{5,2} = P'_{5,2}$. Thus, the first call to `GroundRule` determines $q(3)$ to be true while the second call additionally determines the possible atom $q(2)$.

The grounding of P_6 is illustrated in Tables 4e and 4f. Note that we obtain that x is possible because $p(1)$ was not determined to be true.

The grounding of P_7 is depicted in Tables 4g and 4h. Note that, unlike before, we obtain that y is false because $q(3)$ was determined to be true.

Furthermore, observe that the choice of the refined instantiation sequence determines the output of the algorithm. In fact, swapping $P_{5,1}$ and $P_{5,2}$ in the sequence would result in x being false and y being possible.

Example 22. We illustrate the grounding of aggregates on the company controls example in Tables 1 and 2 using the grounding sequence $(P_i)_{1 \leq i \leq 9}$ and the set of facts F from Example 15. Observe that the grounding of components P_1 to P_8 produces the program $\{a \leftarrow \top \mid a \in F\}$. We focus on how component P_9 is grounded. Because there are no negative dependencies, the components P_9 and P'_9 in Line 5 of Algorithm 3 are equal. To ground component P_9, we use the rewriting from Example 16.

The grounding of component P_9 is illustrated in Table 5, which follows the same conventions as in Example 21. Because the program is positive, the calls in Lines 6 and 7 in Algorithm 3 proceed in the same way and we depict only one of them. Furthermore, because this example involves a recursive rule with an aggregate, the header consists of five rows separated by dashed lines processed by Algorithm 2. The first row corresponds to P_9^ϵ grounded in Line 4, the second and third to P_9^η grounded in Line 5, the fourth to

$0 > \overline{50}$		$c(X)$	$c(Y)$	$X \neq Y$	$\epsilon_{a,r}(X,Y)$	1
$o(X,Y,S)$		$c(X)$	$c(Y)$	$X \neq Y$	$\eta_{e_1,a,r}(S,X,Y)$	2
$s(X,Z)$	$o(Z,Y,S)$	$c(X)$	$c(Y)$	$X \neq Y$	$\eta_{e_2,a,r}(S,Z,X,Y)$	3
`Propagate`						4
$\alpha_{a,r}(X,Y)$		$c(X)$	$c(Y)$	$X \neq Y$	$s(X,Y)$	5
$\times$						1.1
$o(c_1,c_2,60) \longrightarrow c(c_1) \to c(c_2) \to c_1 \neq c_2 \longrightarrow \eta_{e_1,a,r}(60,c_1,c_2)$						1.2
$o(c_1,c_3,20) \longrightarrow c(c_1) \to c(c_3) \to c_1 \neq c_3 \longrightarrow \eta_{e_1,a,r}(20,c_1,c_3)$						
$o(c_2,c_3,35) \longrightarrow c(c_2) \to c(c_3) \to c_2 \neq c_3 \longrightarrow \eta_{e_1,a,r}(35,c_2,c_3)$						
$o(c_3,c_4,51) \longrightarrow c(c_3) \to c(c_4) \to c_3 \neq c_4 \longrightarrow \eta_{e_1,a,r}(51,c_3,c_4)$						
$\times$						1.3
$\{\alpha_{a,r}(c_1,c_2), \alpha_{a,r}(c_3,c_4)\}$						1.4
$\alpha_{a,r}(c_1,c_2) \longrightarrow c(c_1) \to c(c_2) \to c_1 \neq c_2 \longrightarrow s(c_1,c_2)$						1.5
$\alpha_{a,r}(c_3,c_4) \longrightarrow c(c_3) \to c(c_4) \to c_3 \neq c_4 \longrightarrow s(c_3,c_4)$						
$\times$						2.1
$\times$						2.2
$s(c_1,c_2) \to o(c_2,c_3,35) \longrightarrow c(c_1) \to c(c_3) \to c_1 \neq c_3 \to \eta_{e_1,a,r}(35,c_2,c_1,c_3)$						2.3
$s(c_3,c_4) \longrightarrow \times$						
$\{\alpha_{a,r}(c_1,c_3)\}$						2.4
$\alpha_{a,r}(c_1,c_3) \longrightarrow c(c_1) \to c(c_3) \to c_1 \neq c_3 \longrightarrow s(c_1,c_3)$						2.5
$\times$						3.1
$\times$						3.2
$s(c_1,c_3) \to o(c_3,c_4,51) \longrightarrow c(c_1) \to c(c_4) \to c_1 \neq c_4 \to \eta_{e_1,a,r}(51,c_3,c_1,c_4)$						3.3
$\{\alpha_{a,r}(c_1,c_4)\}$						3.4
$\alpha_{a,r}(c_1,c_4) \longrightarrow c(c_1) \to c(c_4) \to c_1 \neq c_4 \longrightarrow s(c_1,c_4)$						3.5
$\times$						4.1
$\times$						4.2
$s(c_1,c_4) \longrightarrow \times$						4.3
$\emptyset$						4.4
$\times$						4.5

TABLE 5. Tracing grounding of component P_9 where c = *company*, o = *owns*, and s = *controls*.

aggregate propagation in Line 6, and the fifth to P_9^α grounded in Line 7. After the header follow the iterations of the loop in Lines 3 to 9. Because the component is recursive, processing the component requires four iterations, which are separated by solid lines in the table. The right-hand-side column of the table contains the iteration number and a number indicating which row in the header is processed. The row for aggregate propagation lists the aggregate atoms that have been propagated.

The grounding of rule r_2 in row 1.1 does not produce any rule instances in any iteration because the comparison $0 > 50$ is false. By first selecting this literal when grounding the rule, the remaining rule body can be completely ignored. Actual systems implement heuristics to prioritize such literals. Next, in the grounding of rule r_3 in row 1.2, direct shares given by facts over *owns*/3 are accumulated. Because the rule does not contain any recursive predicates, it only produces ground instances in the first iteration. Unlike this, rule r_4 contains the recursive predicate *controls*/2. It does not produce instances in the first iteration in row 1.3 because there are no corresponding atoms yet. Next, aggregate propagation is triggered in row 1.4, resulting in aggregate atoms $\alpha_{a,r}(c_1, c_2)$ and $\alpha_{a,r}(c_3, c_4)$, for which enough shares have been accumulated in row 1.2. Note that this corresponds to the propagation of the sets G_1 and G_2 in Example 19. With these atoms, rule r_1 is instantiated in row 1.5, leading to new atoms over *controls*/2. Observe that, by selecting atom $\alpha_{a,r}(X, Y)$ first, `GroundRule` can instantiate the rule without backtracking.

In the second iteration, the newly obtained atoms over *controls*/2 yield atom $\eta_{e_1,a,r}(35, c_2, c_1, c_3)$ in row 2.3, which in turn leads to the aggregate atom $\alpha_{a,r}(c_1, c_3)$ resulting in further instances of r_4. Note that this corresponds to the propagation of the set G_3 in Example 19.

The following iterations proceed in a similar fashion until no new atoms are accumulated and the grounding loop terminates. Note that the utilized selection strategy affects the amount of backtracking in rule instantiation. One particular strategy used in *gringo* is to prefer atoms over recursive predicates. If there is only one such atom, `GroundRule` can select this atom first and only has to consider newly derived atoms for instantiation. The table is made more compact by applying this strategy. Further techniques are available in the database literature [63] that also work in case of multiple atoms over recursive predicates.

7. Refinements

Up to now, we were primarily concerned by characterizing the theoretical and algorithmic cornerstones of grounding. This section refines these concepts by further detailing aggregate propagation, algorithm specifics, and the treatment of language constructs from *gringo*'s input language.

7.1. **Aggregate Propagation**

We used in Section 6 the relative translation of aggregates for propagation, namely, formula $\tau^*_G(a\sigma)$ in Definition 21, to check whether an aggregate is satisfiable. In this section, we identify several aggregate specific properties that allow us to implement more efficient algorithms to perform this check.

To begin with, we establish some properties that greatly simplify the treatment of (arbitrary) monotone or antimonotone aggregates.

We have already seen in Proposition 34 that $\tau^*(a)_I$ is classically equivalent to $\tau^*(a)$ for any closed aggregate a and two-valued interpretation I. Here is its counterpart for antimonotone aggregates.

Proposition 51. *Let a be a closed aggregate.*

If a is antimonotone, then $\tau^*(a)_I$ is classically equivalent to $\top$ if $I \models \tau^*(a)$ and $\bot$ otherwise for any two-valued interpretation I.

See proof on page 92.

Example 23. In Example 19, we check whether the interpretation J satisfies the formulas $\tau^*_{G_1}(a\sigma_1)_I$ to $\tau^*_{G_4}(a\sigma_4)_I$.

Using Proposition 34, this boils down to checking $\sum_{e \in G_i, J \models B(e)} H(e) > 50$ for each $1 \leq i \leq 4$. We get $60 > 50$, $51 > 50$, $55 > 50$, and $35 \not> 50$ for each G_i, which agrees with checking $J \models \tau^*_{G_i}(a\sigma_i)_I$.

An actual implementation can maintain a counter for the current value of the sum for each closed aggregate instance, which can be updated incrementally and compared with the bound as new instances of aggregate elements are grounded.

Next, we see that such counter based implementations are also possible #sum aggregates using the $<$, $\leq$, $>$, or $\geq$ relations. We restrict our attention to finite interpretations because Proposition 52 is intended to give an idea on how to implement an actual propagation algorithm for aggregates (infinite interpretations would add more special cases). Furthermore, we just consider the case that the bound is an integer here; the aggregate is constant for any other ground term.

Proposition 52. *Let I be a finite two-valued interpretation, E be a set of aggregate elements, and b be an integer.*

For $T = H(\{e \in \mathrm{Inst}(E) \mid I \models B(e)\})$, we get

(a) *$\tau^*(\#\mathrm{sum}\{E\} \succ b)_I$ is classically equivalent to $\tau^*(\#\mathrm{sum}^+\{E\} \succ b')$ with $\succ \in \{\geq, >\}$ and $b' = b - \#\mathrm{sum}^-(T)$, and*

(b) *$\tau^*(\#\mathrm{sum}\{E\} \prec b)_I$ is classically equivalent to $\tau^*(\#\mathrm{sum}^-\{E\} \prec b')$ with $\prec \in \{\leq, <\}$ and $b' = b - \#\mathrm{sum}^+(T)$.*

See proof on page 92.

The remaining propositions identify properties that can be exploited when propagating aggregates over the $=$ and $\neq$ relations.

Proposition 53. *Let I be a two-valued interpretation, E be a set of aggregate elements, and b be a ground term.*

We get the following properties:

(a) *$\tau^*(f\{E\} < b)_I \vee \tau^*(f\{E\} > b)_I$ implies $\tau^*(f\{E\} \neq b)_I$, and*

(b) *$\tau^*(f\{E\} = b)_I$ implies $\tau^*(f\{E\} \leq b)_I \wedge \tau^*(f\{E\} \geq b)_I$.*

See proof on page 93.

The following proposition identifies special cases when the implications in Proposition 53 are equivalences. Another interesting aspect of this proposition

is that we can actually replace #sum aggregates over $=$ and $\neq$ with a conjunction or disjunction, respectively, at the expense of calculating a less precise approximate model. The conjunction is even strongly equivalent to the original aggregate under Ferraris' semantics but not the disjunction.

Proposition 54. *Let I and J be two-valued interpretations, f be an aggregate function among* #count, #sum$^+$, #sum$^-$ *or* #sum, *E be a set of aggregate elements, and b be an integer.*

We get the following properties:

(a) *for $I \subseteq J$, we have $J \models \tau^*(f\{E\} < b)_I \vee \tau^*(f\{E\} > b)_I$ iff $J \models \tau^*(f\{E\} \neq b)_I$, and*
(b) *for $J \subseteq I$, we have $J \models \tau^*(f\{E\} = b)_I$ iff $J \models \tau^*(f\{E\} \leq b)_I \wedge \tau^*(f\{E\} \geq b)_I$.*

See proof on page 93.

The following proposition shows that full propagation of #sum, #sum$^+$, or #sum$^-$ aggregates over relations $=$ and $\neq$ involves solving the subset sum problem [51]. We assume that we propagate w.r.t. some polynomial number of aggregate elements. Propagating possible atoms when using the $=$ relation, i.e., when $I \subseteq J$, involves deciding an NP problem and propagating certain atoms when using the $\neq$ relation, i.e., when $J \subseteq I$, involves deciding a co-NP problem.[8] Note that the decision problem for #count aggregates is polynomial, though.

Proposition 55. *Let I and J be finite two-valued interpretations, f be an aggregate function, E be a set of aggregate elements, and b be a ground term.*

For $T_I = \{H(e) \mid e \in \text{Inst}(E), I \models B(e)\}$ and $T_J = \{H(e) \mid e \in \text{Inst}(E), J \models B(e)\}$, we get the following properties:

(a) *for $J \subseteq I$, we have $J \models \tau^*(f\{E\} \neq b)_I$ iff there is no set $X \subseteq T_I$ such that $f(X \cup T_J) = b$, and*
(b) *for $I \subseteq J$, we have $J \models \tau^*(f\{E\} = b)_I$ and iff there is a set $X \subseteq T_J$ such that $f(X \cup T_I) = b$.*

See proof on page 95.

7.2. **Algorithmic Refinements**

The calls in Lines 6 and 7 in Algorithm 3 can sometimes be combined to calculate certain and possible atoms simultaneously. This can be done whenever a component does not contain recursive predicates. In this case, it is sufficient to just calculate possible atoms along with rule instances in Line 7 augmenting Algorithm 1 with an additional check to detect whether a rule instance produces a certain atom. Observe that this condition applies to all stratified components but can also apply to components depending on unstratified components. In fact, typical programs following the generate,

[8]Note that *clingo*'s grounding algorithm does not attempt to solve these problems in all cases. It simply over- or underapproximates the satisfiability using Proposition 53.

define, and test methodology [46, 53] of ASP, where the generate part uses choice rules [56] (see below), do not contain unstratified negation at all. When a grounder is combined with a solver built to store rules and perform inferences, one can optimize for the case that there are no negative recursive predicates in a component. In this case, it is sufficient to compute possible atoms along with their rule instances and leave the computation of certain atoms to the solver. Finally, note that *gringo* currently does not separate the calculation of certain and possible atoms at the expense of computing a less precise approximate model and possibly additional rule instances.

Example 24. For the following example, *gringo* computes atom $p(4)$ as unknown but the algorithms in Section 6 identify it as true.

$$\begin{array}{ll} r(1,4) & p(1) \leftarrow \neg q(1) \\ r(2,3) & q(1) \leftarrow \neg p(1) \\ r(3,1) & p(2) \\ p(Y) \leftarrow p(X) \wedge r(X,Y) & \end{array}$$

When grounding the last rule, *gringo* determines $p(4)$ to be possible in the first iteration because $p(1)$ is unknown at this point. In the second iteration, it detects that $p(1)$ is a fact but does not use it for grounding again. If there were further rules depending negatively on predicate $p/1$, inapplicable rules might appear in *gringo*'s output.

Another observation is that the loop in Algorithm 2 does not produce further rule instances in a second iteration for components without recursive predicates. *Gringo* maintains an index [23] for each positive body literal to speed up matching of literals; whenever none of these indexes, used in rules of the component at hand, are updated, further iterations can be skipped.

Just like *dlv*'s grounder, *gringo* adapts algorithms for semi-naive evaluation from the field of databases. In particular, it works best on linear programs [1], having at most one positive literal occurrence over a recursive predicate in a rule body. The program in Table 1 for the company controls problem is such a linear program because *controls*/2 is the only recursive predicate. Algorithm 1 can easily be adapted to efficiently ground linear programs by making sure that the recursive positive literal is selected first. We then only have to consider matches that induce atoms not already used for instantiations in previous iterations of the loop in Algorithm 2 to reduce the amount of backtracking to find rule instances. In fact, the order in which literals are selected in Line 3 is crucial for the performance of Algorithm 1. *Gringo* uses an adaptation of the selection heuristics presented in [43] that additionally takes into account recursive predicates and terms with function symbols.

To avoid unnecessary backtracking when grounding general logic programs, *gringo* instantiates rules using an algorithm similar to the improved semi-naive evaluation with optimizations for linear rules as in [1].

7.3. **Capturing *gringo*'s input language**

We presented aggregate programs where rule heads are simple atoms. Beyond that, *gringo*'s input language offers more elaborate language constructs to ease modeling.

A prominent such construct are so called choice rules [56]. Syntactically, one-element choice rules have the form $\{a\} \leftarrow B$, where a is an atom and B a body. Semantically, such a rule amounts to $a \vee \neg a \leftarrow B$ or equivalently $a \leftarrow \neg\neg a \wedge B$. We can easily add support for grounding choice rules, that is, rules where the head is not a plain atom but an atom marked as a choice, by discarding choice rules when calculating certain atoms and treating them like normal rules when grounding possible atoms. A translation that allows for supporting head aggregates using a translation to aggregate rules and choice rules is given in [24]. Note that *gringo* implements further refinements to omit deriving head atoms if a head aggregate cannot be satisfied.

Another language feature that can be instantiated in a similar fashion as body aggregates are conditional literals. *Gringo* adapts the rewriting and propagation of body aggregates to also support grounding of conditional literals.

Yet another important language feature are disjunctions in the head of rules [33]. As disjunctive logic programs, aggregate programs allow us to solve problems from the second level of the polynomial hierarchy. In fact, using Łukasiewicz' theorem [50], we can write a disjunctive rule of form

$$a \vee b \leftarrow B$$

as the shifted strongly equivalent $\mathcal{R}$-program:

$$a \leftarrow (b \rightarrow a) \wedge B$$
$$b \leftarrow (a \rightarrow b) \wedge B.$$

We can use this as a template to design grounding algorithms for disjunctive programs. In fact, *gringo* calculates the same approximate model for the disjunctive rule and the shifted program.

The usage of negation as failure is restricted in $\mathcal{R}$-programs. Note that any occurrence of a negated literal l in a rule body can be replaced by an auxiliary atom a adding rule $a \leftarrow l$ to the program. The resulting program preserves the stable models modulo the auxiliary atoms. This translation can serve as a template for double negation or negation in aggregate elements as supported by *gringo*.

Integrity constraints are a straightforward extension of logic programs. They can be grounded just like normal rules deriving an auxiliary atom that stands for $\bot$. Grounding can be stopped whenever the auxiliary atom is derived as certain. Integrity constraints also allow for supporting negated head atoms, which can be shifted to rule bodies [39] resulting in integrity constraints, and then treated like negation in rule bodies.

A frequently used convenience feature of *gringo* are term pools [24, 25]. The grounder handles them by removing them in a rewriting step. For example, a rule of form

$$h(X;Y,Z) \leftarrow p(X;Y), q(Z)$$

is factored out into the following rules:

$$h(X,Z) \leftarrow p(X), q(Z)$$
$$h(X,Z) \leftarrow p(Y), q(Z)$$
$$h(Y,Z) \leftarrow p(X), q(Z)$$
$$h(Y,Z) \leftarrow p(Y), q(Z)$$

We can then apply the grounding algorithms developed in Section 6.

To deal with variables ranging over integers, *gringo* supports interval terms [24, 25]. Such terms are handled by a translation to inbuilt range predicates. For example the program

$$h(l..u)$$

for terms l and u is rewritten into

$$h(A) \leftarrow rng(A,l,u)$$

by introducing auxiliary variable A and range atom $rng(A,l,u)$. The range atom provides matches including all substitutions that assign integer values between l and u to A. Special care has to be taken regarding rule safety, the range atom can only provide bindings for variable A but needs variables in the terms l and u to be provided elsewhere.

A common feature used when writing logic programs are terms involving arithmetic expressions and assignments. Both influence which rules are considered safe by the grounder. For example the rule

$$h(X,Y) \leftarrow p(X+Y,Y) \wedge X = Y + Y$$

is rewritten into

$$h(X,Y) \leftarrow p(A,Y) \wedge X = Y + Y \wedge A = X + Y$$

by introducing auxiliary variable A. The rule is safe because we can match the literals in the order as given in the rewritten rule. The comparison $X = Y + Y$ extends the substitution with an assignment for X and the last comparison serves as a test. *Gringo* does not try to solve complicated equations but supports simple forms like the one given above.

Last but not least, *gringo* does not just support terms in assignments but it also supports aggregates in assignments. To handle such kind of aggregates, the rewriting and propagation of aggregates has to be extended. This is achieved by adding an additional variable to the aggregate replacement atoms (21), which is assigned by propagation. Additional care has to to be taken during the rewriting to ensure that the rewritten rules are safe.

8. Related Work

This section aims at inserting our contributions into the literature, starting with theoretical aspects over algorithmic ones to implementations.

Splitting for infinitary formulas has been introduced in [35] generalizing results of [20, 40]. To this end, the concept of an A-stable models is introduced [35] We obtain the following relationship between our definition of a stable model relative to a set I^{c} and A-stable models: For an $\mathcal{N}$-program P, we have that if X is a stable model of P relative to I^{c}, then $X \cup I^{\mathrm{c}}$ is an $(\mathcal{A} \setminus I^{\mathrm{c}})$-stable model of P. Similarly, we get that if X is an A-stable model of P, then $S_P^{X \setminus A}(X)$ is a stable model of P relative to $X \setminus A$. The difference between the two concepts is that we fix atoms I^{c} in our definition while A-stable models allow for assigning arbitrary truth values to atoms in $\mathcal{A} \setminus A$ (cf. [35, Proposition 1]). With this, let us compare our handling of program sequences to symmetric splitting [35]. Let $(P_i)_{i \in \mathbb{I}}$ be a refined instantiation sequence of aggregate program P, and $F = \bigcup_{i<j} \tau^*(P_i)$ and $G = \bigcup_{i \geq j} \tau^*(P_i)$ for some $j \in \mathbb{I}$ such that $H(F) \neq H(G)$. We can use the infinitary splitting theorem in [35] to calculate the stable model of $F^\wedge \wedge G^\wedge$ through the $H(F)$- and $\mathcal{A} \setminus H(F)$-stable models of $F^\wedge \wedge G^\wedge$. Observe that instantiation sequences do not permit positive recursion between their components and infinite walks are impossible because an aggregate program consists of finitely many rules inducing a finite dependency graph. Note that we additionally require the condition $H(F) \neq H(G)$ because components can be split even if their head atoms overlap. Such a split can only occur if overlapping head atoms in preceding components are not involved in positive recursion.

Next, let us relate our operators to the ones defined in [60]. First of all, it is worthwhile to realize that the motivation in [60] is to conceive operators mimicking model expansion in ID-logic by adding certain atoms. More precisely, let Φ, St, and Wf stand for the versions of the Fitting, stable, and well-founded operators defined in [60]. Then, we get the following relations to the operators defined in the previous sections:

$$\begin{aligned} St_{P,I^{\mathrm{c}}}(J) &= lfp(\Phi_{P,I^{\mathrm{c}}}(\cdot, J)) \\ &= lfp(T_{P,J}^{I^{\mathrm{c}}}) \cup I^{\mathrm{c}} \\ &= S_P^{I^{\mathrm{c}}}(J) \cup I^{\mathrm{c}}. \end{aligned}$$

For the well-founded operator we obtain

$$Wf_{P,I^{\mathrm{c}}}(I, J) = W_P^{I^{\mathrm{c}}, I^{\mathrm{c}}}(I, J) \sqcup I^{\mathrm{c}}.$$

Our operators allow us to directly calculate the atoms derived by a program. The versions in [60] always include the input facts in their output and the well-founded operator only takes certain but not possible atoms as input.

In fact, we use operators as in [13] to approximate the well-founded model and to obtain a ground program. While we apply operators to infinitary formulas (resulting from a translation of aggregates) as introduced in [60],

there has also been work on applying operators directly to aggregates. The authors of [66] provide an overview. Interestingly, the high complexity of approximating the aggregates pointed out in Proposition 55 has already been identified in [54].

Simplification can be understood as a combination of unfolding (dropping rules if a literal in the positive body is not among the head atoms of a program, i.e., not among the possible atoms) and negative reduction (dropping rules if an atom in the negative body is a fact, i.e., the literal is among the certain atoms) [6, 7]. Even the process of grounding can be seen as a directed way of applying unfolding (when matching positive body literals) and negative reduction (when matching negative body literals). When computing facts, only rules whose negative body can be removed using positive reduction are considered.

The algorithms in [42] to calculate well-founded models perform a computation inspired by the alternating sequence to define the well-founded model as in [65]. Our work is different in so far as we are not primarily interested in computing the well-founded model but the grounding of a program. Hence, our algorithms stop after the second application of the stable operator (the first to compute certain and the second to compute possible atoms). At this point, a grounder can use algorithms specialized for propositional programs to simplify the logic program at hand. Algorithmic refinements for normal logic programs as proposed in [42] also apply in our setting.

Last but not least, let us outline the evolution of grounding systems over the last two decades.

The *lparse* [57] grounder introduced domain- or omega-restricted programs [58]. Unlike safety, omega-restrictedness is not modular. That is, the union of two omega-restricted programs is not necessarily omega-restricted while the union of two safe programs is safe. Apart from this, *lparse* supports recursive monotone and antimonotone aggregates. However, our company controls encoding in Table 1 is not accepted because it is not omega-restricted. For example, variable X in the second aggregate element in Table 1 needs a domain predicate. Even if we supplied such a domain predicate, *lparse* would instantiate variable X with all terms provided by the domain predicate resulting in a large grounding. As noted in [21], recursive nonmonotone aggregates (sum aggregates with negative weights) are not supported correctly by *lparse*.

Gringo 1 and 2 add support for lambda-restricted programs [32] extending omega-restricted programs. This augments the set of predicates that can be used for instantiation but is still restricted as compared to safe programs. That is, lambda-restrictedness is also not modular and our company controls program is still not accepted. At the time, the development goal was to be compatible to *lparse* but extend the class of accepted programs. Notably, *gringo* 2 adds support for additional aggregates [28]. Given its origin, *gringo* up to version 4 handles recursive nonmonotone aggregates in the same incorrect way as *lparse*.

The grounder of the *dlv* system has been the first one to implement grounding algorithms based on semi-naive evaluation [15]. Furthermore, it implements various techniques to efficiently ground logic programs [18, 43, 55]. The $dlv^{\mathcal{A}}$ system is the first *dlv*-based system to support recursive aggregates [12], which is nowadays also available in recent versions of *idlv* [10].

Gringo 3 closed up to *dlv* being the first *gringo* version to implement grounding algorithms based on semi-naive evaluation [27]. The system accepts safe rules but still requires lambda-restrictedness for predicates within aggregates. Hence, our company controls encoding is still not accepted.

Gringo 4 implements grounding of aggregates with algorithms similar to the ones presented in Section 6 [29]. Hence, it is the first version that accepts our company controls encoding.

Finally, *gringo* 5 refines the translation of aggregates as in [3] to properly support nonmonotone recursive aggregates and refines the semantics of pools and undefined arithmetics [24].

Another system with a grounding component is the *idp* system [11]. Its grounder instantiates a theory by assigning sorts to variables. Even though it supports inductive definitions, it relies solely on the sorts of variables [67] to instantiate a theory. In case of inductive definitions, this can lead to instances of definitions that can never be applied. We believe that the algorithms presented in Section 6 can also be implemented in an *idp* system decreasing the instantiation size of some problems (e.g., the company controls problem presented in the introduction).

9. Conclusion

We have provided a first comprehensive elaboration of the theoretical foundations of grounding in ASP. This was enabled by the establishment of semantic underpinnings of ASP's modeling language in terms of infinitary (ground) formulas [24, 37]. Accordingly, we start by identifying a restricted class of infinitary programs, namely, $\mathcal{R}$-programs, by limiting the usage of implications. Such programs allow for tighter semantic characterizations than general $\mathcal{F}$-programs, while being expressive enough to capture logic programs with aggregates. Interestingly, we rely on ID-well-founded models [8, 61] to approximate the stable models of $\mathcal{R}$-programs (and simplify them in a stable-models preserving way). This is due do the fact that the ID-well-operator enjoys monotonicity, which lends itself to the characterization of iterative grounding procedures. The actual semantics of non-ground aggregate programs is then defined via a translation to $\mathcal{R}$-programs. This setup allows us to characterize the inner workings of our grounding algorithms for aggregate programs in terms of the operators introduced for $\mathcal{R}$-programs. It turns out that grounding amounts to calculating an approximation of the ID-well-founded model together with a ground program simplified with that model. This does not only allow us prove the correctness of our grounding algorithms but moreover to characterize the output of a grounder like *gringo*

in terms of established formal means. To this end, we have shown how to split aggregate programs into components and to compute their approximate models (and corresponding simplified ground programs). The union of these models corresponds to an approximate model of the whole ground program (and the union of the simplified ground programs corresponds to a simplification of the whole ground program). Even though, we limit ourselves to $\mathcal{R}$-programs, we capture the core aspects of grounding: a monotonically increasing Herbrand base and on-the-fly (ground) rule generation. Additional language features of *gringo*'s input language are relatively straightforward to accommodate by extending the algorithms presented in this paper.

For reference, we implemented the presented algorithms in a prototypical grounder, *μ-gringo*, supporting aggregate programs (see Footnote 1). While it is written to be as concise as possible and not with efficiency in mind, it may serve as a basis for experiments with custom grounder implementations. The actual *gringo* system supports a much larger language fragment. There are some differences compared to the algorithms presented here. First, certain atoms are removed from rule bodies if not explicitly disabled via a command line option. Second, translation τ^* is only used to characterize aggregate propagation. In practice, *gringo* translates ground aggregates to monotone aggregates [3]. Further translation [5] or even native handling [26] of them is left to the solver. Finally, in some cases, *gringo* might produce more rules than the algorithms presented above. This should not affect typical programs. A tighter integration of grounder and solver to further reduce the number of ground rules is an interesting topic of future research.

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Appendix A. Proofs

Proposition 2. *Sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of infinitary formulas are strongly equivalent iff $\mathcal{H}_1^I$ and $\mathcal{H}_2^I$ are classically equivalent for all two-valued interpretations I.*

Proof of Proposition 2. Let I and J be two-valued interpretations, and $\mathcal{H}$ be an infinitary formula. Clearly, $I \models \mathcal{H}^J$ iff $I \cap J \models \mathcal{H}^J$. Thus, we only need to consider interpretations such that $I \subseteq J$. By Lemma 1 in [36], we have that $I \models \mathcal{H}^J$ iff (I, J) is an HT-model of $\mathcal{H}$. The proposition holds because by Theorem 3 Item (iii) in [36], we have that $\mathcal{H}_1$ and $\mathcal{H}_2$ are strongly equivalent iff they have the same HT models. □

Proposition 3. *The $\mathcal{F}$-program P has the same stable models as the formula $\{B(r) \to H(r) \mid r \in P\}^\wedge$.*

Proof of Proposition 3. Let $F = \{B(r) \to H(r) \mid r \in P\}^\wedge$.

First, we consider the case $I \models F$. By Lemma 1 in [60], this implies that P^I and F^I are classically equivalent and thus have the same minimal models.[9] Thus, I is a stable model of P iff I is a stable model of F.

Second, we consider the case $I \not\models F$ and show that I is neither a stable model of F nor P. Proposition 1 in [60] states that I is a model of F iff I is a model of F^I. Thus, I is not a stable model of F. Furthermore, because $I \not\models F$, there is a rule $r \in P$ such that $I \not\models B(r) \to H(r)$. Consequently, we have $I \models B(r)$ and $I \not\models H(r)$. Using the above proposition again, we get $I \models B(r)^I$. Because $I \models B(r)^I$ and $I \not\models H(r)$, we get $I \not\models r^I$ and in turn $I \not\models P^I$. Thus, I is not a stable model of P either. □

Proposition 4. *Let F be a formula, and I and J be interpretations.*
If F is positive and $I \subseteq J$, then $I \models F$ implies $J \models F$.

Proof of Proposition 4. This property can be shown by induction over the rank of the formula. □

Proposition 5. *Let F be a formula, and I and J be interpretations.*
Then,

(a) *if F is positive then F^I is positive,*
(b) *$I \models F$ iff $I \models F^I$,*
(c) *if F is strictly positive and $I \subseteq J$ then $I \models F$ iff $I \models F^J$.*

Proof of Proposition 5.

Property (a). Because the reduct only replaces subformulas by $\bot$, the resulting formula is still positive.

Property (b). Corresponds to Proposition 1 in [60].

Property (c). This property can be shown by induction over the rank of the formula. □

[9] To be precise, Lemma 1 in [60] is stated for a set of formulas, which can be understood as an infinitary conjunction.

Proposition 6. *Let F be a formula, and I, J, and X be interpretations. Then,*

(a) *F_I is positive,*
(b) *$I \models F$ iff $I \models F_I$,*
(c) *if F is positive then $F = F_I$, and*
(d) *if $I \subseteq J$ then $X \models F_J$ implies $X \models F_I$.*

Proof of Proposition 6.

Property (a). Because the ID-reduct replaces all negative occurrences of atoms, the resulting formula is positive.

Property (b). This property is easy to see because if reduct the replaces an atom a, it is replaced with either $\top$ or $\bot$ depending on whether $I \models a$ or $I \not\models a$. This does not change the satisfaction of the subformula w.r.t. I.

Property (c). Because a positive formula does not contain negative occurrences of atoms, it is not changed by the ID-reduct.

Property (d). We prove by induction over the rank of formula F that

$$X \models F_J \text{ implies } X \models F_I \text{ and} \tag{24}$$

$$X \models F_{\bar{I}} \text{ implies } X \models F_{\bar{J}}. \tag{25}$$

Base. We consider the case that F is a formula of rank 0.

$$F \text{ is a formula of rank 0 implies } F \text{ is an atom.}$$

First, we show (24). We assume $X \models F_J$:

$$F \text{ is an atom implies } F_I = F_J = F.$$
$$F_I = F_J \text{ and } X \models F_J \text{ implies } X \models F_I.$$

Second, we show (25). We assume $X \models F_{\bar{I}}$:

$$F \text{ is an atom and } X \models F_{\bar{I}} \text{ implies } F_{\bar{I}} = \top.$$
$$F_{\bar{I}} = \top \text{ implies } F \in I.$$
$$F \in I \text{ and } I \subseteq J \text{ implies } F \in J.$$
$$F \text{ is an atom and } F \in J \text{ implies } F_{\bar{J}} = \top.$$
$$F_{\bar{J}} = \top \text{ implies } X \models F_{\bar{J}}.$$

Hypothesis. We assume that (24) and (25) hold for formulas F of ranks smaller than i.

Step. We only show (24) because (25) can be shown in a similar way. We consider formulas F of rank i.

First, we consider the case that F is a conjunction of form $\mathcal{H}^{\wedge}$.

$$X \models \mathcal{H}^{\wedge}_J \text{ implies } X \models G_J \text{ for all } G \in \mathcal{H}.$$
$$X \models G_J \text{ implies } X \models G_I \text{ by Hypothesis.}$$
$$X \models G_I \text{ implies } X \models \mathcal{H}^{\wedge}_I.$$

The case for disjunctions can be proven in a similar way.

Last, we consider the case that F is an implication of form $G \rightarrow H$. Observe that

$$F_I = G_{\bar{I}} \rightarrow H_I \text{ and}$$
$$F_J = G_{\bar{J}} \rightarrow H_J.$$

First, we consider the case $X \not\models G_{\bar{J}}$:

$$X \not\models G_{\bar{J}} \text{ implies } X \not\models G_{\bar{I}} \text{ by Hypothesis.}$$
$$X \not\models G_{\bar{I}} \text{ implies } X \models F_I.$$

Second, we consider the case $X \models H_J$:

$$X \models H_J \text{ implies } X \models H_I \text{ by Hypothesis.}$$
$$X \models H_I \text{ implies } X \models F_I. \qquad \square$$

Lemma 10. *Let P be an $\mathcal{F}$-Program.*

Then, the well-founded model $WM(P)$ of P is consistent.

Proof of Lemma 10. This lemma follows from Proposition 14 in [13] observing that the well-founded operator is a monotone symmetric operator. The proposition is actually a bit more general stating that the operator maps any consistent four-valued interpretation to a consistent four-valued interpretation. $\square$

Lemma 56. *Let O and O' be monotone operators over complete lattice $(L, \leq)$ with $O'(x) \leq O(x)$ for each $x \in L$.*

Then, we get $x' \leq x$ where x' and x are the least fixed points of O' and O, respectively.

Proof of Lemma 56. Let y be a prefixed point of O. We have $O(y) \leq y$. Because $O'(y) \leq O(y)$, we get $O'(y) \leq y$. So each prefixed point of O is also a prefixed point of O'.

Let S' and S be the set of all prefixed points of O' and O, respectively. We obtain $S \subseteq S'$. By Theorem 1 (a), we get that x' is the greatest lower bound of S'. Observe that x' is a lower bound for S. By construction of S, we have $x \in S$. Hence, we get $x' \leq x$. $\square$

Lemma 57. *Let P and P' be $\mathcal{F}$-programs and I be an interpretation.*

Then, $P' \subseteq P$ implies $S_{P'}(I) \subseteq S_P(I)$.

Proof of Lemma 57. This lemma is a direct consequence of Lemma 56 observing that the one-step provability operator derives fewer consequences for P'. $\square$

Lemma 58. *Let P be an $\mathcal{R}$-program, I be a two-valued interpretation, and $J = S_P(I)$.*

Then, X is a stable model of P, $I \subseteq X$, and $I \subseteq J$ implies $X \subseteq J$.

Proof of Lemma 58. Because X is a stable model of P, it is the only minimal model of P^X. Furthermore, we have that J is a model of P_I. To show that $X \subseteq J$, we show that J is also a model of P^X. For this, it is enough to show that for each rule $r \in P$ we have $J \not\models B(r)_I$ implies $J \not\models B(r)^X$. We prove inductively over the rank of the formula $F = B(r)$ that $J \not\models F_I$ implies $J \not\models F^X$.

Base. We consider the case that F is a formula of rank 0.

If $X \not\models F$, we get $J \not\models F^X$ because $F^X = \bot$. Thus, we only have to consider the case $X \models F$:

$$
\begin{aligned}
F \text{ is a formula of rank 0 implies } & F \text{ is an atom.} \\
F \text{ is an atom implies } & F_I = F. \\
X \models F \text{ and } F \text{ is an atom implies } & F^X = F. \\
F_I = F \text{ and } F^X = F \text{ implies } & F_I = F^X. \\
J \not\models F_I \text{ and } F_I = F^X \text{ implies } & J \not\models F^X.
\end{aligned}
$$

Hypothesis. We assume that $J \not\models F_I$ implies $J \not\models F^X$ holds for formulas F of ranks smaller than i.

Step. We consider the case that F is a formula of rank i.

As in the base case, we only have to consider the case $X \models F$. Furthermore, we have to distinguish the cases that F is a conjunction, disjunction, or implication.

We first consider the case that F is a conjunction of form $\mathcal{F}^\wedge$:

$$
\begin{aligned}
X \models F \text{ implies } & F^X = \{G^X \mid G \in \mathcal{F}\}^\wedge. \\
J \not\models F_I \text{ and } F_I = \{G_I \mid G \in \mathcal{F}\}^\wedge \text{ implies } & J \not\models G_I \text{ for some } G \in \mathcal{F}. \\
G \in \mathcal{F} \text{ and } F \text{ has rank } i \text{ implies } & G \text{ has rank less than } i. \\
J \not\models G_I \text{ and } G \text{ has rank less than } i \text{ implies } & J \not\models G^X \text{ by Hypothesis.} \\
J \not\models G^X \text{ and } F^X = \{G^X \mid G \in F\}^\wedge \text{ implies } & J \not\models F^X.
\end{aligned}
$$

The case that F is a disjunction can be shown in a similar way to the case that F is a conjunction.

Last, we consider the case that F is an implication of form $G \to H$. Observe that G is positive because F has no occurrences of implications in its antecedent and, furthermore, given that F is a formula of rank i, H is a formula of rank less than i.

We show $I \models G$:

$$
\begin{aligned}
J \not\models F_I \text{ and } F_I = G_{\bar{I}} \to H_I \text{ implies } & J \models G_{\bar{I}} \\
J \models G_{\bar{I}} \text{ and } G \text{ is positive implies } & I \models G \text{ because } G_{\bar{I}} \equiv \top.
\end{aligned}
$$

We show $J \models G^X$:

G is positive, $I \subseteq X$, and $I \models G$ implies $X \models G$ by Proposition 4.
$X \models F$, $X \models G$, and $F = G \to H$ implies $X \models H$.
G is positive, $I \subseteq X$, and $I \models G$ implies $I \models G^X$ by Proposition 5 (c).
G is positive implies G^X is positive by Proposition 5 (a).
G^X is positive, $I \subseteq J$, and $I \models G^X$ implies $J \models G^X$ by Proposition 4.

We show $J \not\models H^X$:

$I \models G$ and $F_I = G_{\bar{I}} \to H_I$ implies $F_I \equiv H_I$ because $G_{\bar{I}} \equiv \top$.
$F_I \equiv H_I$ and $J \not\models F_I$ implies $J \not\models H_I$.
$J \not\models H_I$ and H has rank less than i implies $J \not\models H^X$ by hypothesis.

Because $X \models F$, we have $F^X = G^X \to H^X$. Using $J \models G^X$ and $J \not\models H^X$, we get $J \not\models F^X$. □

Theorem 11. *Let P be an $\mathcal{R}$-program and (I, J) be the well-founded model of P.*
If X is a stable model of P, then $I \subseteq X \subseteq J$.

Proof of Theorem 11. Let X be a stable model of P.

We prove by transfinite induction over the sequence of postfixed points leading to the well-founded model:

$$\begin{aligned}(I_0, J_0) &= (\emptyset, \Sigma),\\ (I_{\alpha+1}, J_{\alpha+1}) &= W_P(I_\alpha, J_\alpha) \text{ for ordinals } \alpha, \text{ and}\\ (I_\beta, J_\beta) &= (\bigcup_{\alpha<\beta} I_\alpha, \bigcap_{\alpha<\beta} J_\alpha) \text{ for limit ordinals } \beta.\end{aligned}$$

We have that $\alpha < \beta$ implies $(I_\alpha, J_\alpha) \leq_p (I_\beta, J_\beta)$ for ordinals α and β, $I_\alpha \subseteq J_\alpha$ for ordinals α, and there is a least ordinal α such that $(I, J) = (I_\alpha, J_\alpha)$.

Base. We have $I_0 \subseteq X \subseteq J_0$.

Hypothesis. We assume $I_\beta \subseteq X \subseteq J_\beta$ for all ordinals $\beta < \alpha$.

Step. If $\alpha = \beta + 1$ is a successor ordinal we have

$$\begin{aligned}(I_\alpha, J_\alpha) &= W_P(I_\beta, J_\beta)\\ &= (S_P(J_\beta), S_P(I_\beta)).\end{aligned}$$

By the induction hypothesis we have $I_\beta \subseteq X \subseteq J_\beta$.

First, we show $I_\alpha \subseteq X$:

$$\begin{aligned} X \text{ is a (stable) model implies } & S_P(X) \subseteq X. \\ X \subseteq J_\beta \text{ implies } & S_P(J_\beta) \subseteq S_P(X). \\ S_P(X) \subseteq X \text{ and } S_P(J_\beta) \subseteq S_P(X) \text{ implies } & S_P(J_\beta) \subseteq X. \\ I_\alpha = S_P(J_\beta) \text{ and } S_P(J_\beta) \subseteq X \text{ implies } & I_\alpha \subseteq X. \end{aligned}$$

Second, we show $X \subseteq J_\alpha$:

$$\begin{aligned} \beta < \alpha \text{ implies } & (I_\beta, J_\beta) \leq_p (I_\alpha, J_\alpha) \\ (I_\beta, J_\beta) \leq_p (I_\alpha, J_\alpha) \text{ and } I_\alpha \subseteq J_\alpha \text{ implies } & I_\beta \subseteq J_\alpha \\ X \text{ is a stable model, } I_\beta \subseteq X, & \\ J_\alpha = S_P(I_\beta), \text{ and } I_\beta \subseteq J_\alpha \quad & \text{implies } X \subseteq J_\alpha \text{ by Lemma 58.} \end{aligned}$$

We have shown $I_\alpha \subseteq X \subseteq J_\alpha$ for successor ordinals.

If α is a limit ordinal we have

$$(I_\alpha, J_\alpha) = (\bigcup_{\beta<\alpha} I_\beta, \bigcap_{\beta<\alpha} J_\beta).$$

Let $x \in I_\alpha$. There must be an ordinal $\beta < \alpha$ such that $x \in I_\beta$. Since $I_\beta \subseteq X$ by the hypothesis, we have $x \in X$. Thus, $I_\alpha \subseteq X$.

Let $x \in X$. For each ordinal $\beta < \alpha$ we have $x \in J_\beta$ because $X \subseteq J_\beta$ by the hypothesis. Thus, we get $x \in J_\alpha$. It follows that $X \subseteq J_\alpha$.

We have shown $I_\alpha \subseteq X \subseteq J_\alpha$ for limit ordinals. □

Lemma 12. *Let P be an $\mathcal{F}$-program and (I, J) be a four-valued interpretation.*

Then, we have $H(P^{I,J}) = T_{P_I}(J)$.

Proof of Lemma 12. The program $P^{I,J}$ contains all rules $r \in P$ such that $J \models B(r)_I$. This are the exactly the rules whose heads are gathered by the T operator. □

Proposition 13. *Let P be an $\mathcal{F}$-program and (I, J) be the well-founded model of P.*

Then, we have

(a) *$S_{P^{I,J}}(I') = J$ for all $I' \subseteq I$, and*
(b) *$S_{P^{I,J}}(J') = S_P(J')$ for all $J \subseteq J'$.*

Proof of Proposition 13. Throughout the proof we use

$$\begin{aligned} S_P(J) &= I, \\ S_P(I) &= J, \\ P^{I,J} &\subseteq P, \text{ and} \\ I &\subseteq J \text{ because the well-founded model is consistent.} \end{aligned}$$

Property (a). We show $J = S_{P^{I,J}}(I)$. Let $\hat{J} = S_{P^{I,J}}(I)$ and $r \in P \setminus P^{I,J}$:

$$
\begin{aligned}
P^{I,J} \subseteq P \text{ and} \\
\hat{J} = S_{P^{I,J}}(I) \text{ and} \\
J = S_P(I) \text{ implies } \hat{J} \subseteq J \text{ by Lemma 57.} \\
r \notin P^{I,J} \text{ implies } J \not\models B(r)_I. \\
\hat{J} \subseteq J \text{ and } J \not\models B(r)_I \text{ implies } \hat{J} \not\models B(r)_I \text{ by Proposition 4.} \\
\hat{J} = S_{P^{I,J}}(I) \text{ and } \hat{J} \not\models B(r)_I \text{ implies } \hat{J} \models P_I. \\
\hat{J} \models P_I \text{ and } J = S_P(I) \text{ implies } J \subseteq \hat{J}. \\
J \subseteq \hat{J} \text{ and } \hat{J} \subseteq J \text{ implies } J = \hat{J}.
\end{aligned}
$$

Thus, we get that $S_{P^{I,J}}(I) = J$.

With this we can continue to prove $S_{P^{I,J}}(I') = J$. Let $r \in P^{I,J}$:

$$
\begin{aligned}
r \in P^{I,J} \text{ implies } J \models B(r)_I. \\
r \in P^{I,J} \text{ and } P^{I,J} \subseteq P \text{ implies } r \in P. \\
J \models B(r)_I,\ r \in P,\ \text{and } S_P(I) = J \text{ implies } H(r) \in J. \\
J \models B(r)_I \text{ and } I' \subseteq I \text{ implies } J \models B(r)_{I'} \text{ by Proposition 6 (d).} \\
H(r) \in J \text{ and } J \models B(r)_{I'} \text{ implies } S_{P^{I,J}}(I') \subseteq J. \\
I' \subseteq I \text{ and } J = S_{P^{I,J}}(I) \text{ implies } J \subseteq S_{P^{I,J}}(I').
\end{aligned}
$$

Thus, we get $S_{P^{I,J}}(I') = J$.

Property (b). Let $I' = S_{P^{I,J}}(J')$ and $r \in P \setminus P^{I,J}$:

$$
\begin{aligned}
r \notin P^{I,J} \text{ implies } J \not\models B(r)_I. \\
I \subseteq J,\ J \subseteq J',\ \text{and } J \not\models B(r)_I \text{ implies } J \not\models B(r)_{J'}. \text{ by Proposition 6 (d)..} \\
I' \subseteq I,\ I \subseteq J,\ \text{and } J \not\models B(r)_{J'} \text{ implies } I' \not\models B(r)_{J'}. \text{ by Proposition 4.} \\
I' = S_{P^{I,J}}(J') \text{ and } I' \not\models B(r)_{J'} \text{ implies } S_P(J') \subseteq S_{P^{I,J}}(J'). \\
P^{I,J} \subseteq P \text{ implies } S_{P^{I,J}}(J') \subseteq S_P(J') \text{ by Lemma 57.}
\end{aligned}
$$

Thus, we get $S_{P^{I,J}}(J') = S_P(J')$. □

Theorem 14. *Let P be an $\mathcal{F}$-program and (I, J) be the well-founded model of P.*

Then, P and $P^{I,J}$ have the same well-founded model.

Proof of Theorem 14. By Proposition 13, we have $(I, J) = W_{P^{I,J}}(I, J)$. Let $(\widehat{I}, \widehat{J}) = WM(P^{I,J})$:

$$(\widehat{I}, \widehat{J}) = WM(P^{I,J}) \text{ and } (I, J) = W_{P^{I,J}}(I, J) \text{ implies } (\widehat{I}, \widehat{J}) \leq_p (I, J).$$
by Theorem 1 (c).

$$\widehat{I} \subseteq I \text{ implies } S_{P^{I,J}}(\widehat{I}) = S_{P^{I,J}}(I)$$
by Proposition 13 (a).

$$\hat{J} = S_{P^{I,J}}(\widehat{I}) = S_{P^{I,J}}(I) = J \text{ implies } \widehat{J} = J.$$

$$\widehat{I} = S_{P^{I,J}}(\widehat{J}),\ S_{P^{I,J}}(J) = I, \text{ and } \widehat{J} = J \text{ implies } \widehat{I} = I.$$

We obtain $(I, J) = (\widehat{I}, \widehat{J})$. □

Theorem 15. *Let P be an $\mathcal{F}$-program, and I, J and X be two-valued interpretations.*

If $I \subseteq X \subseteq J$, then X is a stable model of P iff X is a stable model of $P^{I,J}$.

Proof of Theorem 15. We first show that all rule bodies removed by the simplification are falsified by X. Let $r \in P \setminus P^{I,J}$ and assume $X \models B(r)$:

$$X \models B(r) \text{ implies } X \models B(r)_X \text{ by Proposition 6 (b).}$$

$$X \models B(r)_X \text{ and } I \subseteq X \text{ implies } X \models B(r)_I \text{ by Proposition 6 (d).}$$

$$X \models B(r)_I \text{ and } X \subseteq J \text{ implies } J \models B(r)_I \text{ by Proposition 4.}$$

This is a contradiction and, thus, $X \not\models B(r)$. We use the following consequence in the proof below:

$$X \not\models B(r) \text{ implies } (P \setminus P^{I,J})^X \equiv \emptyset.$$

To show the theorem, we show that P^X and $(P^{I,J})^X$ have the same minimal models. Clearly, we have $P^X = (P^{I,J})^X \cup (P \setminus P^{I,J})^X$. Using this and $(P \setminus P^{I,J})^X \equiv \emptyset$, we obtain that P^X and $(P^{I,J})^X$ have the same minimal models. □

Corollary 16. *Let P be an $\mathcal{R}$-program and (I, J) be the well-founded model of P.*

Then, P and $P^{I,J}$ have the same stable models.

Proof of Corollary 16. The result follows from Theorems 11, 14 and 15. □

Theorem 17. *Let P and Q be $\mathcal{F}$-programs, and (I, J) be the well-founded model of P.*

If $P^{I,J} \subseteq Q \subseteq P$, then P and Q have the same well-founded models.

Proof of Theorem 17. By Lemma 57, we have $S_{P^{I,J}}(X) \subseteq S_Q(X) \subseteq S_P(X)$ for any two-valued interpretation X. Thus, by Theorem 14, we get $(I, J) = W_Q(I, J)$.

Next, let $(\widehat{I},\widehat{J})$ be a prefixed point of W_Q with $(\widehat{I},\widehat{J}) \leq_p (I,J)$. We have $(S_Q(\widehat{J}), S_Q(\widehat{I})) \leq_p (\widehat{I},\widehat{J}) \leq_p (I,J)$.

$$J \subseteq \widehat{J} \text{ implies } S_{P^{I,J}}(\widehat{J}) = S_Q(\widehat{J}) = S_P(\widehat{J})$$
by Proposition 13 (b).

$$S_Q(\widehat{J}) = S_P(\widehat{J}) \text{ and } S_Q(\widehat{J}) \subseteq \widehat{I} \text{ implies } S_P(\widehat{J}) \subseteq \widehat{I}.$$

$$\widehat{J} \subseteq S_Q(\widehat{I}) \text{ and } S_Q(\widehat{I}) \subseteq S_P(\widehat{I}) \text{ implies } \widehat{J} \subseteq S_P(\widehat{I}).$$

$$S_P(\widehat{J}) \subseteq \widehat{I} \text{ and } \widehat{J} \subseteq S_P(\widehat{I}) \text{ implies } W_P(\widehat{I},\widehat{J}) \leq_p (\widehat{I},\widehat{J}).$$

$$W_P(\widehat{I},\widehat{J}) \leq_p (\widehat{I},\widehat{J}) \text{ implies } (I,J) \leq_p (\widehat{I},\widehat{J})$$
by Theorem 1 (a).

$$(I,J) \leq_p (\widehat{I},\widehat{J}) \text{ and } (\widehat{I},\widehat{J}) \leq_p (I,J) \text{ implies } (I,J) = (\widehat{I},\widehat{J}).$$

By Theorem 1 (a), we obtain that $WM(Q) = (I,J)$. □

Corollary 18. *Let P and Q be $\mathcal{R}$-programs, and (I,J) be the well-founded model of P.*

If $P^{I,J} \subseteq Q \subseteq P$, then P and Q are equivalent.

Proof of Corollary 18. Observe that $P^{I,J} = Q^{I,J}$. With this, the corollary follows from Corollary 16 and Theorem 17. □

Proposition 20. *Let P be an $\mathcal{F}$-program, and I^c and J be two-valued interpretations.*

We get the following properties:

(a) *$J' \subseteq J$ implies $S_P^{I^c}(J) \subseteq S_P^{I^c}(J')$, and*

(b) *$I^{c\prime} \subseteq I^c$ implies $S_P^{I^{c\prime}}(J) \subseteq S_P^{I^c}(J)$.*

Proof of Proposition 20. Both properties can be shown by inspecting the reduced programs.

Property (a). Observe that we can equivalently write $S_P^{I^c}(J) = S_{pe_{I^c}(P)}(J)$ and $S_P^{I^c}(J') = S_{pe_{I^c}(P)}(J')$. With this and Proposition 8, we see that the relative stable operator is antimonotone just as the stable operator.

Property (b). Observe that $S_P^{I^c}(J)$ is equal to the least fixed point of $T_{P_J}^{I^c}$ and $S_P^{I^{c\prime}}(J)$ is equal to the least fixed point of $T_{P_J}^{I^{c\prime}}$. Furthermore, observe that $T_{P_J}^{I^{c\prime}}(X) \subseteq T_{P_J}^{I^c}(X)$ for any two-valued interpretation X because $I^{c\prime} \subseteq I^c$ and the underlying T operator is monotone. With this and Lemma 56, we have shown the property. □

Proposition 22. *Let P be an $\mathcal{F}$-program, and (I,J) and (I^c,J^c) be four-valued interpretations.*

We get the following properties:

(a) *$(I',J') \leq_p (I,J)$ implies $W_P^{I^c,J^c}(I',J') \leq_p W_P^{I^c,J^c}(I,J)$, and*

(b) *$(I^{c\prime},J^{c\prime}) \leq_p (I^c,J^c)$ implies $W_P^{I^{c\prime},J^{c\prime}}(I,J) \leq_p W_P^{I^c,J^c}(I,J)$.*

Proof of Proposition 22. Both properties can be shown by using the monotonicity of the underlying relative ID-stable operator:

Property (a). Given that $S_P^{I^c}$ is antimonotone and $J' \cup J^c \subseteq J \cup J^c$, we have $S_P^{I^c}(J \cup J^c) \subseteq S_P^{I^c}(J' \cup J^c)$. Analogously, we can show $S_P^{J^c}(I' \cup I^c) \subseteq S_P^{J^c}(I \cup I^c)$. We get $(S_P^{I^c}(J \cup J^c), S_P^{J^c}(I \cup I^c)) \leq_p (S_P^{I^c}(J' \cup J^c), S_P^{J^c}(I' \cup I^c))$.

Hence, $W_P^{I^c,J^c}$ is monotone.

Property (b). We have to show $(S_P^{I^{c\prime}}(J \cup J^{c\prime}), S_P^{J^{c\prime}}(I \cup I^{c\prime})) \leq_p (S_P^{I^c}(J \cup J^c), S_P^{J^c}(I \cup I^c))$.

Given that $I^{c\prime} \subseteq I^c$ and $J \cup J^c \subseteq J \cup J^{c\prime}$, we obtain $S_P^{I^{c\prime}}(J \cup J^{c\prime}) \subseteq S_P^{I^c}(J \cup J^c)$ using Proposition 20. The same argument can be used for the possible atoms of the four-valued interpretations. Given that $J^c \subseteq J^{c\prime}$ and $I \cup I^{c\prime} \subseteq I \cup I^c$, we obtain $S_P^{J^c}(I \cup I^c) \subseteq S_P^{J^{c\prime}}(I \cup I^{c\prime})$ using Proposition 20.

Hence, we have shown $W_P^{I^{c\prime},J^{c\prime}}(I,J) \leq_p W_P^{I^c,J^c}(I,J)$. □

Observation 59. *Let P be an $\mathcal{F}$-program, and I, I' and I^c be two-valued interpretations.*

We get the following properties:

(a) *$I \models P$ and $I^c \subseteq I$ implies $I \models pe_{I^c}(P)$,*
(b) *$I \models pe_{I^c}(P)$ and $I' \cap B(P)^+ \subseteq I^c$ implies $I \cup I' \models pe_{I^c}(P)$, and*
(c) *$I \models pe_{I^c}(P)$ implies $I \models P$.*

Proposition 23. *Let P^b and P^t be $\mathcal{F}$-programs, I^c and J be two-valued interpretations, $I = S_{P^b \cup P^t}^{I^c}(J)$, $I^e = I \cap (B(P^b)^+ \cap H(P^t))$, $I^b = S_{P^b}^{I^c \cup I^e}(J)$, and $I^t = S_{P^t}^{I^c \cup I^b}(J)$.*

Then, we have $I = I^b \cup I^t$.

Proof of Proposition 23. Let $\widetilde{I} = I^b \cup I^t$. Furthermore, we use the following programs:

$$\widehat{P}^b = pe_{I^c}(P_J^b) \qquad \widetilde{P}^b = pe_{I^c \cup I^e}(P_J^b) = pe_{I^e}(\widehat{P}^b)$$
$$\widehat{P}^t = pe_{I^c}(P_J^t) \qquad \widetilde{P}^t = pe_{I^c \cup I^b}(P_J^t) = pe_{I^b}(\widehat{P}^t)$$

Observe that

$$I = S_{P^b \cup P^t}^{I^c}(J) = LM(\widehat{P}^b \cup \widehat{P}^t),$$
$$I^b = S_{P^b}^{I^c \cup I^e}(J) = LM(\widetilde{P}^b), \text{ and}$$
$$I^t = S_{P^t}^{I^c \cup I^b}(J) = LM(\widetilde{P}^t).$$

To show that $\widetilde{I} \subseteq I$, we show that I is a model of both $\widetilde{P}^b$ and $\widetilde{P}^t$. To show that $I \subseteq \widetilde{I}$, we show that $\widetilde{I}$ is a model of both $\widehat{P}^b$ and $\widehat{P}^t$.

Property $I \models \widetilde{P}^{\mathrm{b}}$.

$$I = LM(\widehat{P}^{\mathrm{b}} \cup \widehat{P}^{\mathrm{t}}) \text{ implies } I \models \widehat{P}^{\mathrm{b}}.$$
$$I \models \widehat{P}^{\mathrm{b}} \text{ and } I^{\mathrm{e}} \subseteq I \text{ implies } I \models \widetilde{P}^{\mathrm{b}}$$
$$\text{by Observation 59 (a).}$$

Property $I \models \widetilde{P}^{\mathrm{t}}$.

$$I = LM(\widehat{P}^{\mathrm{b}} \cup \widehat{P}^{\mathrm{t}}) \text{ implies } I \models \widehat{P}^{\mathrm{t}}.$$
$$I \models \widetilde{P}^{\mathrm{b}} \text{ and } I^{\mathrm{b}} = LM(\widetilde{P}^{\mathrm{b}}) \text{ implies } I^{\mathrm{b}} \subseteq I.$$
$$I \models \widehat{P}^{\mathrm{t}} \text{ and } I^{\mathrm{b}} \subseteq I \text{ implies } I \models \widetilde{P}^{\mathrm{t}}.$$
$$\text{by Observation 59 (a).}$$

Property $\widetilde{I} \models \widehat{P}^{\mathrm{b}}$. Let $E = B(P^{\mathrm{b}})^{+} \cap H(P^{\mathrm{t}})$:

$$\widetilde{I} \subseteq I \text{ and } I^{\mathrm{e}} = I \cap E \text{ implies } I^{\mathrm{t}} \cap E \subseteq I^{\mathrm{e}}.$$
$$I^{\mathrm{t}} = LM(\widetilde{P}^{\mathrm{t}}) \text{ implies } I^{\mathrm{t}} \subseteq H(\widetilde{P}^{\mathrm{t}}).$$
$$I^{\mathrm{t}} \cap E \subseteq I^{\mathrm{e}} \text{ and } I^{\mathrm{t}} \subseteq H(\widetilde{P}^{\mathrm{t}}) \text{ implies } I^{\mathrm{t}} \cap B(P^{\mathrm{b}})^{+} \subseteq I^{\mathrm{e}}.$$
$$I^{\mathrm{t}} \cap B(P^{\mathrm{b}})^{+} \subseteq I^{\mathrm{e}} \text{ implies } I^{\mathrm{t}} \cap B(\widehat{P}^{\mathrm{b}})^{+} \subseteq I^{\mathrm{e}}.$$
$$I^{\mathrm{t}} \cap B(\widehat{P}^{\mathrm{b}})^{+} \subseteq I^{\mathrm{e}} \text{ and } I^{\mathrm{b}} = LM(\widetilde{P}^{\mathrm{b}}) \text{ implies } \widetilde{I} \models \widetilde{P}^{\mathrm{b}}$$
$$\text{by Observation 59 (b).}$$
$$\widetilde{I} \models \widetilde{P}^{\mathrm{b}} \text{ implies } \widetilde{I} \models \widehat{P}^{\mathrm{b}}$$
$$\text{by Observation 59 (c).}$$

Property $\widetilde{I} \models \widehat{P}^{\mathrm{t}}$.

$$I^{\mathrm{t}} = LM(\widetilde{P}^{\mathrm{t}}) \text{ implies } \widetilde{I} \models \widetilde{P}^{\mathrm{t}}$$
$$\text{by Observation 59 (b).}$$
$$\widetilde{I} \models \widetilde{P}^{\mathrm{t}} \text{ implies } \widetilde{I} \models \widehat{P}^{\mathrm{t}}$$
$$\text{by Observation 59 (c).} \qquad \square$$

Proposition 24. *Let* P^{b} *and* P^{t} *be* $\mathcal{F}$*-programs,* $(I^{\mathrm{c}}, J^{\mathrm{c}})$ *be a four-valued interpretation,* $(I, J) = WM^{I^{\mathrm{c}},J^{\mathrm{c}}}(P^{\mathrm{b}} \cup P^{\mathrm{t}})$, $(I^{\mathrm{e}}, J^{\mathrm{e}}) = (I, J) \sqcap (B(P^{\mathrm{b}})^{\pm} \cap H(P^{\mathrm{t}}))$, $(I^{\mathrm{b}}, J^{\mathrm{b}}) = WM^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(I^{\mathrm{e}},J^{\mathrm{e}})}(P^{\mathrm{b}})$, *and* $(I^{\mathrm{t}}, J^{\mathrm{t}}) = WM^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(I^{\mathrm{b}},J^{\mathrm{b}})}(P^{\mathrm{t}})$.

Then, we have $(I, J) = (I^{\mathrm{b}}, J^{\mathrm{b}}) \sqcup (I^{\mathrm{t}}, J^{\mathrm{t}})$.

Proof of Proposition 24. Let $P = P^{\mathrm{b}} \cup P^{\mathrm{t}}$ and $E = B(P^{\mathrm{b}})^{\pm} \cap H(P^{\mathrm{t}})$. We begin by evaluating P, P^{b} and P^{t} w.r.t. (I, J) and obtain

$$\begin{aligned}(I, J) &= W_P^{I^{\mathrm{c}},J^{\mathrm{c}}}(I, J)\\ &= (S_P^{I^{\mathrm{c}}}(J^{\mathrm{c}} \cup J), S_P^{J^{\mathrm{c}}}(I^{\mathrm{c}} \cup I)),\end{aligned}$$

$$\begin{aligned}(\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) &= W_{P^{\mathrm{b}}}^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(I^{\mathrm{e}},J^{\mathrm{e}})}(I,J)\\ &= (S_{P^{\mathrm{b}}}^{I^{\mathrm{c}}\cup I^{\mathrm{e}}}(J^{\mathrm{c}} \cup J^{\mathrm{e}} \cup J), S_{P^{\mathrm{b}}}^{J^{\mathrm{c}}\cup J^{\mathrm{e}}}(I^{\mathrm{c}} \cup I^{\mathrm{e}} \cup I)), \text{ and}\\ (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) &= W_{P^{\mathrm{t}}}^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(\widehat{I}^{\mathrm{b}},\widehat{J}^{\mathrm{b}})}(I,J)\\ &= (S_{P^{\mathrm{t}}}^{I^{\mathrm{c}}\cup \widehat{I}^{\mathrm{b}}}(J^{\mathrm{c}} \cup \widehat{J}^{\mathrm{b}} \cup J), S_{P^{\mathrm{t}}}^{J^{\mathrm{c}}\cup \widehat{J}^{\mathrm{b}}}(I^{\mathrm{c}} \cup \widehat{I}^{\mathrm{b}} \cup I)).\end{aligned}$$

Using $(I^{\mathrm{e}}, J^{\mathrm{e}}) \sqsubseteq (I, J)$ we get

$$(\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) = (S_{P^{\mathrm{b}}}^{I^{\mathrm{c}}\cup I^{\mathrm{e}}}(J^{\mathrm{c}} \cup J), S_{P^{\mathrm{b}}}^{J^{\mathrm{c}}\cup J^{\mathrm{e}}}(I^{\mathrm{c}} \cup I)).$$

By Proposition 23 and Observation 21 (c), we get

$$\begin{aligned}(\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) &\sqsubseteq (I, J),\\ (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) &= (S_{P^{\mathrm{t}}}^{I^{\mathrm{c}}\cup I^{\mathrm{e}}}(J^{\mathrm{c}} \cup J), S_{P^{\mathrm{t}}}^{J^{\mathrm{c}}\cup J^{\mathrm{e}}}(I^{\mathrm{c}} \cup I)), \text{ and}\\ (I, J) &= (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqcup (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}).\end{aligned}$$

We first show $(I^{\mathrm{b}}, J^{\mathrm{b}}) = (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}})$ and then $(I^{\mathrm{t}}, J^{\mathrm{t}}) = (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}})$.

Property $(I^{\mathrm{b}}, J^{\mathrm{b}}) \leq_p (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}})$.

$$(\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqsubseteq (I, J) \text{ and}$$
$$(I^{\mathrm{e}}, J^{\mathrm{e}}) = (I, J) \sqcap E \text{ implies } (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}}) \sqsubseteq (I, J).$$
$$(I, J) = (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqcup (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \text{ implies } (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqsubseteq (I, J).$$
$$(\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqsubseteq H(P^{\mathrm{t}}) \text{ implies } (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqcap E.$$
$$(\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqcap E \text{ and}$$
$$(\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqsubseteq (I, J) \text{ implies } (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (I^{\mathrm{e}}, J^{\mathrm{e}}).$$
$$(\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (I^{\mathrm{e}}, J^{\mathrm{e}}) \text{ and}$$
$$(I, J) = (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqcup (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \text{ implies } (I, J) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}}).$$

With the above, we use Observation 21 (c) to show that $(\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}})$ is a fixed point of $W_{P^{\mathrm{b}}}^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(J^{\mathrm{c}},J^{\mathrm{e}})}$:

$$\begin{aligned}(\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) &= W_{P^{\mathrm{b}}}^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(I^{\mathrm{e}},J^{\mathrm{e}})}(I,J)\\ &= W_{P^{\mathrm{b}}}^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(I^{\mathrm{e}},J^{\mathrm{e}})}((I,J) \sqcap B(P^{\mathrm{b}})^{\pm})\\ &= W_{P^{\mathrm{b}}}^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(I^{\mathrm{e}},J^{\mathrm{e}})}((\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}}))\\ &= W_{P^{\mathrm{b}}}^{(I^{\mathrm{c}},J^{\mathrm{c}})\sqcup(I^{\mathrm{e}},J^{\mathrm{e}})}(\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}})\end{aligned}$$

Thus, by Theorem 1 (c), $(I^{\mathrm{b}}, J^{\mathrm{b}}) \leq_p (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}})$.

Property $(I^{\mathrm{b}}, J^{\mathrm{b}}) = (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}})$. To show the property, let

$$(\widetilde{I}, \widetilde{J}) = (I^{\mathrm{b}}, J^{\mathrm{b}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}}) \sqcup (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}),$$

$$(\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) = W_P^{I^{\mathrm{c}}, J^{\mathrm{c}}}(\widetilde{I}, \widetilde{J}) \sqcap E,$$
$$(\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) = (S_{P^{\mathrm{b}}}^{I^{\mathrm{c}} \cup \widetilde{I}^{\mathrm{e}}}(J^{\mathrm{c}} \cup \widetilde{J}), S_{P^{\mathrm{b}}}^{J^{\mathrm{c}} \cup \widetilde{J}^{\mathrm{e}}}(I^{\mathrm{c}} \cup \widetilde{I})),$$
$$(\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) = (S_{P^{\mathrm{t}}}^{I^{\mathrm{c}} \cup \widetilde{I}^{\mathrm{b}}}(J^{\mathrm{c}} \cup \widetilde{J}), S_{P^{\mathrm{t}}}^{J^{\mathrm{c}} \cup \widetilde{J}^{\mathrm{b}}}(I^{\mathrm{c}} \cup \widetilde{I})), \text{ and}$$
$$W_P^{I^{\mathrm{c}}, J^{\mathrm{c}}}(\widetilde{I}, \widetilde{J}) = (\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \sqcup (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) \text{ by Proposition 23.}$$

We get:

$$(I^{\mathrm{b}}, J^{\mathrm{b}}) \leq_p (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \text{ implies } (\widetilde{I}, \widetilde{J}) \leq_p (I, J).$$
$$(\widetilde{I}, \widetilde{J}) \leq_p (I, J) \text{ implies } W_P^{I^{\mathrm{c}}, J^{\mathrm{c}}}(\widetilde{I}, \widetilde{J}) \leq_p (I, J).$$
$$W_P^{I^{\mathrm{c}}, J^{\mathrm{c}}}(\widetilde{I}, \widetilde{J}) \leq_p (I, J) \text{ implies } (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) \leq_p (I^{\mathrm{e}}, J^{\mathrm{e}}).$$
$$(\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (I^{\mathrm{e}}, J^{\mathrm{e}}) \text{ implies } (\widetilde{I}, \widetilde{J}) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}}).$$
$$(\widetilde{I}, \widetilde{J}) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \sqcup (I^{\mathrm{e}}, J^{\mathrm{e}}) \text{ implies } \widetilde{I}^{\mathrm{b}} = S_{P^{\mathrm{b}}}^{I^{\mathrm{c}} \cup \widetilde{I}^{\mathrm{e}}}(J^{\mathrm{c}} \cup J^{\mathrm{e}} \cup J^{\mathrm{b}})$$
$$\text{and} \quad \widetilde{J}^{\mathrm{b}} = S_{P^{\mathrm{b}}}^{J^{\mathrm{c}} \cup \widetilde{J}^{\mathrm{e}}}(I^{\mathrm{c}} \cup I^{\mathrm{e}} \cup I^{\mathrm{b}}).$$
$$\text{by Observation 21 (c).}$$
$$\widetilde{I}^{\mathrm{b}} = S_{P^{\mathrm{b}}}^{I^{\mathrm{c}} \cup \widetilde{I}^{\mathrm{e}}}(J^{\mathrm{c}} \cup J^{\mathrm{e}} \cup J^{\mathrm{b}}) \text{ and}$$
$$\widetilde{J}^{\mathrm{b}} = S_{P^{\mathrm{b}}}^{J^{\mathrm{c}} \cup \widetilde{J}^{\mathrm{e}}}(I^{\mathrm{c}} \cup I^{\mathrm{e}} \cup I^{\mathrm{b}}) \text{ and}$$
$$I^{\mathrm{b}} = S_{P^{\mathrm{b}}}^{I^{\mathrm{c}} \cup I^{\mathrm{e}}}(J^{\mathrm{c}} \cup J^{\mathrm{e}} \cup J^{\mathrm{b}}) \text{ and}$$
$$J^{\mathrm{b}} = S_{P^{\mathrm{b}}}^{J^{\mathrm{c}} \cup J^{\mathrm{e}}}(I^{\mathrm{c}} \cup I^{\mathrm{e}} \cup I^{\mathrm{b}}) \text{ and}$$
$$(\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) \leq_p (I^{\mathrm{e}}, J^{\mathrm{e}}) \text{ implies } (\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \leq_p (I^{\mathrm{b}}, J^{\mathrm{b}})$$
$$\text{by Proposition 22 (b).}$$
$$(\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \leq_p (I^{\mathrm{b}}, J^{\mathrm{b}}) \text{ and } (I^{\mathrm{b}}, J^{\mathrm{b}}) \leq_p (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \text{ implies } (\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \leq_p (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}).$$
$$\widehat{I}^{\mathrm{t}} = S_{P^{\mathrm{t}}}^{I^{\mathrm{c}} \cup \widehat{I}^{\mathrm{b}}}(J^{\mathrm{c}} \cup J) \text{ and}$$
$$\widehat{J}^{\mathrm{t}} = S_{P^{\mathrm{t}}}^{J^{\mathrm{c}} \cup \widehat{J}^{\mathrm{b}}}(I^{\mathrm{c}} \cup I) \text{ and}$$
$$\widetilde{I}^{\mathrm{t}} = S_{P^{\mathrm{t}}}^{I^{\mathrm{c}} \cup \widetilde{I}^{\mathrm{b}}}(J^{\mathrm{c}} \cup \widetilde{J}) \text{ and}$$
$$\widetilde{J}^{\mathrm{t}} = S_{P^{\mathrm{t}}}^{J^{\mathrm{c}} \cup \widetilde{J}^{\mathrm{b}}}(I^{\mathrm{c}} \cup \widetilde{I}) \text{ and}$$
$$(\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \leq_p (\widehat{I}^{\mathrm{b}}, \widehat{J}^{\mathrm{b}}) \text{ and}$$
$$(\widetilde{I}, \widetilde{J}) \leq_p (I, J) \text{ implies } (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) \leq_p (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}})$$
$$\text{by Proposition 20 (a) and (b).}$$
$$W_P^{I^{\mathrm{c}}, J^{\mathrm{c}}}(\widetilde{I}, \widetilde{J}) = (\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \sqcup (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) \text{ and}$$
$$(\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \leq_p (I^{\mathrm{b}}, J^{\mathrm{b}}) \text{ and } (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) \leq_p (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) \text{ implies } W_P^{I^{\mathrm{c}}, J^{\mathrm{c}}}(\widetilde{I}, \widetilde{J}) \leq_p (I^{\mathrm{b}}, J^{\mathrm{b}}) \sqcup (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}).$$
$$(\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) \leq_p (I^{\mathrm{e}}, J^{\mathrm{e}}) \text{ and}$$
$$(\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) \sqsubseteq W_P^{I^{\mathrm{c}}, J^{\mathrm{c}}}(\widetilde{I}, \widetilde{J}) \text{ and}$$

$$
\begin{aligned}
W_P^{I^c,J^c}(\widetilde{I},\widetilde{J}) \leq_p (I^b,J^b) \sqcup (\widehat{I}^t,\widehat{J}^t) &\text{ and}\\
(\widetilde{I},\widetilde{J}) = (I^b,J^b) \sqcup (I^e,J^e) \sqcup (\widehat{I}^t,\widehat{J}^t) &\text{ implies } W_P^{I^c,J^c}(\widetilde{I},\widetilde{J}) \leq_p (\widetilde{I},\widetilde{J}).\\
WM^{I^c,J^c}(P) = (I,J) &\text{ and}\\
W_P^{I^c,J^c}(\widetilde{I},\widetilde{J}) \leq_p (\widetilde{I},\widetilde{J}) &\text{ implies } (I,J) \leq_p (\widetilde{I},\widetilde{J})\\
&\text{by Theorem 1 (a).}\\
(\widetilde{I},\widetilde{J}) \leq_p (I,J) \text{ and } (I,J) \leq_p (\widetilde{I},\widetilde{J}) &\text{ implies } (I,J) = (\widetilde{I},\widetilde{J}).\\
(\widetilde{I},\widetilde{J}) = (I^b,J^b) \sqcup (I^e,J^e) \sqcup (\widehat{I}^t,\widehat{J}^t) &\text{ implies } (I^b,J^b) \sqcup (I^e,J^e) \sqsubseteq (\widetilde{I},\widetilde{J}).\\
(\widetilde{I},\widetilde{J}) = (I^b,J^b) \sqcup (I^e,J^e) \sqcup (\widehat{I}^t,\widehat{J}^t) &\text{ and}\\
(\widehat{I}^t,\widehat{J}^t) \sqcap B(P^b)^{\pm} \sqsubseteq (I^e,J^e) &\text{ implies } (\widetilde{I},\widetilde{J}) \sqcap B(P^b)^{\pm} \sqsubseteq (I^b,J^b) \sqcup (I^e,J^e).\\
(I,J) = (\widetilde{I},\widetilde{J}) &\text{ and}\\
(I^b,J^b) \sqcup (I^e,J^e) \sqsubseteq (\widetilde{I},\widetilde{J}) &\text{ and}\\
(\widetilde{I},\widetilde{J}) \sqcap B(P^b)^{\pm} \sqsubseteq (I^b,J^b) \sqcup (I^e,J^e) &\text{ and}\\
(I^b,J^b) = W_{P^b}^{(I^c,J^c)\sqcup(I^e,J^e)}(I^b,J^b) &\text{ and}\\
(\widehat{I}^b,\widehat{J}^b) = W_{P^b}^{(I^c,J^c)\sqcup(I^e,J^e)}(I,J) &\text{ implies } (I^b,J^b) = (\widehat{I}^b,\widehat{J}^b)\\
&\text{by Observation 21 (c).}
\end{aligned}
$$

Property $(\widehat{I}^t,\widehat{J}^t) = (I^t,J^t)$. Observe that the lemma can be applied with P^b and P^t exchanged. Let

$$
\begin{aligned}
\widetilde{E} &= B(P^t)^{\pm} \cap H(P^b),\\
(\widetilde{I}^e,\widetilde{J}^e) &= (I,J) \cap \widetilde{E},\\
(\widetilde{I}^t,\widetilde{J}^t) &= WM^{(I^c,J^c)\sqcup(\widetilde{I}^e,\widetilde{J}^e)}(P^t), \text{ and}\\
(\widetilde{I}^b,\widetilde{J}^b) &= W_{P^b}^{(I^c,J^c)\sqcup(\widetilde{I}^t,\widetilde{J}^t)}(I,J).
\end{aligned}
$$

Using the properties shown so far, we obtain

$$(I,J) = (\widetilde{I}^t,\widetilde{J}^t) \sqcup (\widetilde{I}^b,\widetilde{J}^b).$$

With this we get:

$$
\begin{aligned}
(I^b,J^b) = (\widehat{I}^b,\widehat{J}^b) &\text{ and}\\
(I,J) = (\widehat{I}^b,\widehat{J}^b) \sqcup (\widehat{I}^t,\widehat{J}^t) &\text{ and}\\
(I^b,J^b) \sqsubseteq H(P^b) &\text{ and}\\
(\widetilde{I}^e,\widetilde{J}^e) = (I,J) \sqcap \widetilde{E} &\text{ implies } (I^b,J^b) \sqcap B(P^t)^{\pm} \sqsubseteq (\widetilde{I}^e,\widetilde{J}^e).\\
(I,J) = (\widetilde{I}^t,\widetilde{J}^t) \sqcup (\widetilde{I}^b,\widetilde{J}^b) &\text{ and}\\
(\widetilde{I}^b,\widetilde{J}^b) \sqsubseteq H(P^b) &\text{ and}
\end{aligned}
$$

$$(\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) = (I, J) \sqcap \widetilde{E} \text{ implies } (\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \sqcap B(P^{\mathrm{t}})^{\pm} \sqsubseteq (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}).$$

$$(\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \sqcap B(P^{\mathrm{t}})^{\pm} \sqsubseteq (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) \text{ and}$$

$$(I, J) = (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) \sqcup (\widetilde{I}^{\mathrm{b}}, \widetilde{J}^{\mathrm{b}}) \text{ implies } (I, J) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) \sqcup (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}).$$

$$(I^{\mathrm{b}}, J^{\mathrm{b}}) \sqcap B(P^{\mathrm{t}})^{\pm} \sqsubseteq (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) \text{ and}$$

$$(I, J) \sqcap B(P^{\mathrm{b}})^{\pm} \sqsubseteq (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) \sqcup (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) \text{ and}$$

$$(\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) = W_{P^{\mathrm{t}}}^{(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}})}(\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) \text{ and}$$

$$(\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) = W_{P^{\mathrm{t}}}^{(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{b}}, J^{\mathrm{b}})}(I, J) \text{ implies } (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) = (\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}})$$
by Observation 21 (c).

$$(I^{\mathrm{b}}, J^{\mathrm{b}}) \sqcap B(P^{\mathrm{t}})^{\pm} \sqsubseteq (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}}) \text{ and}$$

$$(\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) = WM^{(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (\widetilde{I}^{\mathrm{e}}, \widetilde{J}^{\mathrm{e}})}(P^{\mathrm{t}}) \text{ and}$$

$$(I^{\mathrm{t}}, J^{\mathrm{t}}) = WM^{(I^{\mathrm{c}}, J^{\mathrm{c}}) \sqcup (I^{\mathrm{b}}, J^{\mathrm{b}})}(P^{\mathrm{t}}) \text{ implies } (\widetilde{I}^{\mathrm{t}}, \widetilde{J}^{\mathrm{t}}) = (I^{\mathrm{t}}, J^{\mathrm{t}})$$
by Observation 21 (c).

Thus, we get $(\widehat{I}^{\mathrm{t}}, \widehat{J}^{\mathrm{t}}) = (I^{\mathrm{t}}, J^{\mathrm{t}})$. □

Theorem 25. *Let $(P_i)_{i \in \mathbb{I}}$ be a sequence of $\mathcal{F}$-programs.*
Then, $WM((P_i)_{i \in \mathbb{I}}) \leq_p WM(\bigcup_{i \in \mathbb{I}} P_i)$.

Proof of Theorem 25. The theorem can be shown by transfinite induction over the sequence indices. We do not give the full induction proof here but focus on the key idea. Let (I'_i, J'_i) be the intermediate interpretations as in (5) when computing the well-founded model of the sequence. Furthermore, let

$$(I_i, J_i) = WM^{(I_i^{\mathrm{c}}, J_i^{\mathrm{c}}) \sqcup (I_i^{\mathrm{e}}, J_i^{\mathrm{e}})}(P_i)$$

be the intermediate interpretations where $(I_i^{\mathrm{c}}, J_i^{\mathrm{c}})$ is the union of the intermediate interpretations as in (4) and

$$(I_i^{\mathrm{e}}, J_i^{\mathrm{e}}) = (I, J) \cap E_i$$

with E_i as in (3).

Observe that with Proposition 24, we have $WM(\bigcup_{i \in \mathbb{I}} P_i) = \bigcup_{i \in \mathbb{I}} (I_i, J_i)$. By Proposition 22 (b), we have $(I'_i, J'_i) \leq_p (I_i, J_i)$ and, thus, we obtain that $WM((P_i)_{i \in \mathbb{I}}) \leq_p WM(\bigcup_{i \in \mathbb{I}} P_i)$. □

Corollary 26. *Let $(P_i)_{i \in \mathbb{I}}$ be a sequence of $\mathcal{F}$-programs and E_i defined as in (3).*
If $E_i = \emptyset$ for all $i \in \mathbb{I}$ then $WM((P_i)_{i \in \mathbb{I}}) = WM(\bigcup_{i \in \mathbb{I}} P_i)$.

Proof of Corollary 26. This can be proven in the same way as Theorem 25 but note that because E_i is empty, we get $(I'_i, J'_i) = (I_i, J_i)$. □

Lemma 27. *Let P be an $\mathcal{F}$-program such that $B(P)^- \cap H(P) = \emptyset$ and (I^c, J^c) be a four-valued interpretation.*

Then, $WM^{I^c,J^c}(P) = (S_P^{I^c}(J^c), S_P^{J^c}(I^c))$.

Proof of Lemma 27. Let $(I, J) = WM^{I^c,J^c}(P)$.

We have $J = S_P^{J^c}(I^c \cup I)$. By Observation 21 (b), we get $J \subseteq H(P)$. With this and $B(P)^- \cap H(P) = \emptyset$, we get $B(P)^- \cap J = \emptyset$. Thus, $S_P^{I^c}(J^c \cup J) = S_P^{I^c}(J^c)$ by Observation 21 (c).

The same arguments apply to show $S_P^{J^c}(I^c \cup I) = S_P^{J^c}(I^c)$. □

Theorem 28. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{F}$-programs, $(I, J) = WM((P_i)_{i\in\mathbb{I}})$, and E_i, (I_i^c, J_i^c), and (I_i, J_i) be defined as in (3) to (5).*

Then, $P_k^{I,J} \subseteq P_k^{(I_k^c,J_k^c)\sqcup(I_k,J_k)\sqcup(\emptyset,E_k)} \subseteq P_k$ for all $k \in \mathbb{I}$.

Proof of Theorem 28. By Theorem 25, we have

$$\bigsqcup_{i\in\mathbb{I}}(I_i, J_i) \leq_p (I, J).$$

We get $\bigcup_{i\in\mathbb{I}} I_i \subseteq I$ and thus

$$\bigcup_{i\leq k} I_i = I_k^c \cup I_k \subseteq I.$$

Using $J \subseteq \bigcup_{i\in\mathbb{I}} J_i$ and $J_i \subseteq H(P_i)$, we get

$$\begin{aligned} J \cap B(P_k)^{\pm} &\subseteq (\bigcup_{i\leq k} J_i \cup \bigcup_{k<i} H(P_i)) \cap B(P_k)^{\pm} \\ &\subseteq (\bigcup_{i\leq k} J_i \cup E_k) \cap B(P_k)^{\pm} \\ &\subseteq (J_k^c \cup J_k \cup E_k) \cap B(P_k)^{\pm}. \end{aligned}$$

Using both results, we obtain

$$((I_k^c, J_k^c) \sqcup (\emptyset, E_k) \sqcup (I_k, J_k)) \sqcap B(P_k)^{\pm} \leq_p (I, J) \sqcap B(P_k)^{\pm}.$$

Because the body literals determine the simplification, we get

$$P_k^{I,J} \subseteq P_k^{(I_k^c,J_k^c)\sqcup(\emptyset,E_k)\sqcup(I_k,J_k)}.$$ □

Corollary 29. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{F}$-programs, $(I, J) = WM((P_i)_{i\in\mathbb{I}})$, and E_i, (I_i^c, J_i^c), and (I_i, J_i) be defined as in (3) to (5).*

If $E_i = \emptyset$ for all $i \in \mathbb{I}$, then $P_k^{I,J} = P_k^{(I_k^c,J_k^c)\sqcup(I_k,J_k)}$ for all $k \in \mathbb{I}$.

Proof of Corollary 29. This can be proven in the same way as Theorem 28 but note that because E_i is empty, all $\leq_p$ and most $\subseteq$ relations can be replaced with equivalences. □

Corollary 30. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{R}$-programs, and (I, J) be the well-founded model of $\bigcup_{i\in\mathbb{I}} P_i$.*

Then, $\bigcup_{i\in\mathbb{I}} P_i$ and $\bigcup_{i\in\mathbb{I}} Q_i$ with $P_i^{I,J} \subseteq Q_i \subseteq P_i$ have the same well-founded and stable models.

Proof of Corollary 30. This corollary is a direct consequence of Theorems 17 and 28 and Corollary 18. □

Corollary 31. *Let $(P_i)_{i\in\mathbb{I}}$ be a sequence of $\mathcal{R}$-programs, and E_i, (I_i^c, J_i^c), and (I_i, J_i) be defined as in (3) to (5).*

Then, $\bigcup_{i\in\mathbb{I}} P_i$ and $\bigcup_{i\in\mathbb{I}} P_i^{(I_i^c,J_i^c)\sqcup(I_i,J_i)\sqcup(\emptyset,E_i)}$ have the same well-founded and stable models.

Proof of Corollary 31. This corollary is a direct consequence of Theorem 28 and Corollary 30. □

Proposition 32 ([37])**.**

- *Aggregates over functions $\#\mathrm{sum}^+$ and $\#\mathrm{count}$ together with aggregate relations $>$ and $\geq$ are monotone.*
- *Aggregates over functions $\#\mathrm{sum}^+$ and $\#\mathrm{count}$ together with aggregate relations $<$ and $\leq$ are antimonotone.*
- *Aggregates over function $\#\mathrm{sum}^-$ have the same monotonicity properties as $\#\mathrm{sum}^+$ aggregates with the complementary relation.*

Proposition 33. *Let a be a closed aggregate.*

Then, $\tau(a)$ and $\tau^*(a)$ are strongly equivalent.

Proof of Proposition 33. We use Proposition 2 to show that both formulas are strongly equivalent.

Property $I \models \tau^*(a)^J$ implies $I \models \tau(a)^J$ for arbitrary interpretations I. The formulas $\tau^*(a)$ and $\tau(a)$ only differ in the consequents of their implications. Observe that the consequents in $\tau^*(a)$ are stronger than the ones in $\tau(a)$. Thus, it follows that $\tau^*(a)$ is stronger than $\tau(a)$. Furthermore, observe that the same holds for their reducts.

Property $I \not\models \tau^*(a)^J$ implies $I \not\models \tau(a)^J$ for arbitrary interpretations I. Let G be the set of all instance of the aggregate elements of a. Because $I \not\models \tau^*(a)^J$, there must be a set $D \subseteq G$ such that $D \not\models a$, $I \models (\tau(D)^\wedge)^J$, and $I \not\models (\tau_a^*(D)^\vee)^J$. With this, we construct the set

$$\widehat{D} = D \cup \{e \in G \setminus D \mid I \models \tau(e)^J\}.$$

The construction of $\widehat{D}$ and

$$D \not\models a \text{ and}$$

$$I \not\models (\tau_a^*(D)^\vee)^J \text{ implies } \widehat{D} \not\models a \text{ and } I \not\models (\tau_a(\widehat{D})^\vee)^J.$$

The construction of $\widehat{D}$ and

$$I \models (\tau(D)^\wedge)^J \text{ implies } I \models (\tau(\widehat{D})^\wedge)^J.$$

$$\widehat{D} \not\models a \text{ and}$$

$$I \not\models (\tau_a(\widehat{D})^\vee)^J \text{ and}$$

$$I \models (\tau(\widehat{D})^\wedge)^J \text{ implies } I \not\models \tau(a)^J. \qquad \square$$

Proposition 34. *Let a be a closed aggregate.*

If a is monotone, then $\tau^*(a)_I$ is classically equivalent to $\tau^*(a)$ for any two-valued interpretation I.

Proof of Proposition 34. Let G be the set of ground instances of the aggregate elements of a. Furthermore, observe that a monotone aggregate a is either constantly true or not justified by the empty set.

In case that $\tau^*(a) \equiv \top$, we get $\tau^*(a)_I \equiv \top$ and the lemma holds.

Next, we consider the case that the empty set does not justify the aggregate. Observe that $\tau_a^*(\emptyset)$ is stronger than $\tau_a^*(D)$ for any $D \subseteq G$. And, we have that $\tau^*(a)$ contains the implication $\top \rightarrow \tau_a^*(\emptyset)$. Because of this, we have $\tau^*(a) \equiv \tau_a^*(\emptyset)$. Furthermore, all consequents in $\tau^*(a)$ are positive formulas and, thus, not modified by the reduct. Thus, the reduct $(\top \rightarrow \tau_a^*(\emptyset))_I$ is equal to $\top \rightarrow \tau_a^*(\emptyset)$. And as before, it is stronger than all other implications in $\tau^*(a)_I$. Hence, we get $\tau^*(a)_I \equiv \tau_a^*(\emptyset)$. $\square$

Proposition 35. *Let a be a closed aggregate, and $I \subseteq J$ and $X \subseteq J$ be two-valued interpretations.*

Then,

(a) $X \models \tau^*(a)$ *iff* $X \models \tau_J^*(a)$,
(b) $X \models \tau^*(a)_I$ *iff* $X \models \tau_J^*(a)_I$, *and*
(c) $X \models \tau^*(a)^I$ *iff* $X \models \tau_J^*(a)^I$.

Proof of Proposition 35. Remember that the translation $\tau^*(a)$ is a conjunction of implications. The antecedents of the implications are conjunctions of aggregate elements and the consequents are disjunctions of conjunctions of aggregate elements.

Property (a). If the conjunction in an antecedent contains an element not in J, then the conjunction is not satisfied by X and the implication does not affect the satisfiability of $\tau^*(a)$. If a conjunction in a consequent contains an element not in J, then X does not satisfy the conjunction and the conjunction does not affect the satisfiability of the encompassing disjunction. Observe that both cases correspond exactly to those subformulas omitted in $\tau_J^*(a)$.

The remaining two properties follow for similar reasons. □

Lemma 37. *Let P be an aggregate program and $(P_i)_{i\in\mathbb{I}}$ be an instantiation sequence for P.*

Then, for the sequence $(G_i)_{i\in\mathbb{I}}$ with $G_i = \tau^*(P_i)$, we have $E_i = \emptyset$ for each $i \in \mathbb{I}$ where E_i is defined as in (3).

Proof of Lemma 37. This lemma is a direct consequence of Observation 36 (a) and the anti-symmetry of the dependency relation between components. □

Lemma 38. *Let P be an aggregate program and $(P_i)_{i\in\mathbb{I}}$ be an instantiation sequence for P.*

Then, for the sequence $(G_i)_{i\in\mathbb{I}}$ with $G_i = \tau^*(P_i)$, we have $I_i = J_i = S_{G_i}^{I_i^c}(I_i^c)$ for each stratified component P_i where (I_i^c, J_i^c) and (I_i, J_i) are defined as in (4) *and* (5) *in the construction of the well-founded model of $(G_i)_{i\in\mathbb{I}}$ in Definition 7.*

Proof of Lemma 38. In the following, we use E_i and (I_i^c, J_i^c) for the sequence $(G_i)_{i\in\mathbb{I}}$ as defined in (3) and (4). Note that, by Lemma 37, we have $E_i = \emptyset$.

We prove by induction.

Base. Let P_i be a stratified component that does not depend on any other component. Because P_i does not depend on any other component, we have $\bigcup_{j<i} H(G_j) \cap B(G_i)^{\pm} = \emptyset$. Thus, by Observation 21 (b), we get $I_i^c \cap B(G_i)^{\pm} = J_i^c \cap B(G_i)^{\pm} = \emptyset$. By Observation 21 (c), we get $(I_i, J_i) = WM^{I_i^c, J_i^c}(G_i) = WM^{I_i^c, I_i^c}(G_i)$. Because P_i is stratified, we have $B(G_i)^- \cap H(G_i) = \emptyset$. We then use Lemma 27 to obtain $I_i = J_i = S_{G_i}^{I_i^c}(I_i^c)$.

Hypothesis. We assume that the theorem holds for any component P_j with $j < i$.

Step. Let P_i be a stratified component. For any $j < i$, component P_i either depends on P_j or not. If P_i depends on P_j, then P_j is stratified and we get $I_j = J_j$, by the induction hypothesis. If P_i does not depend on P_j, then $I_j \cap B(G_i)^{\pm} = J_j \cap B(G_i)^{\pm} = \emptyset$. By Observation 21 (c), we get $(I_i, J_i) = WM^{I_i^c, J_i^c}(G_i) = WM^{I_i^c, I_i^c}(G_i)$. Just as in the base case, by Lemma 27, we get $I_i = J_i = S_{G_i}^{I_i^c}(I_i^c)$. □

Lemma 39. *Let P be an aggregate program and $(P_{i,j})_{(i,j)\in\mathbb{J}}$ be a refined instantiation sequence for P.*

Then, for the sequence $(G_{i,j})_{(i,j)\in\mathbb{J}}$ with $G_{i,j} = \tau^*(P_{i,j})$, we have $E_{i,j} \cap B(G_{i,j})^+ = \emptyset$ for each $(i,j) \in \mathbb{J}$ where $E_{i,j}$ is defined as in (3).

Proof of Lemma 39. The same arguments as in the proof of Lemma 37 can be used but using Observation 36 (b) instead. □

Lemma 40. *Let P be an aggregate program, E be a set of predicates, and (I^c, J^c) be a four-valued interpretation.*

If $\mathrm{pred}(H(P)) \cap \mathrm{pred}(B(P)^-) \subseteq E$ *then* $AM_E^{I^c, J^c}(P) \leq_p WM^{I^c, J^c \cup E^c}(\tau^*(P))$ *where E^c is the set of all ground atoms over predicates in E.*

Proof of Lemma 40. Let $G = \tau^*(P)$, $G' = \tau^*(P')$ with P' as in Definition 14, $(I, J) = AM_E^{I^c,J^c}(P)$, and $(I', J') = WM^{I^c,J^c\cup E^c}(G)$.

We first show $I \subseteq I'$, or equivalently

$$S_{G'}^{I^c}(J^c) \subseteq S_G^{I^c}(J^c \cup E^c \cup J').$$

Because, $G' \subseteq G$ we get

$$S_{G'}^{I^c}(J^c \cup E^c \cup J') \subseteq S_G^{I^c}(J^c \cup E^c \cup J').$$

Because $\mathrm{pred}(B(P')^-) \cap E = \emptyset$, all rules $r \in G'$ satisfy $B(r)^- \cap E^c \neq \emptyset$ and we obtain

$$S_{G'}^{I^c}(J^c \cup J') \subseteq S_{G'}^{I^c}(J^c \cup E^c \cup J').$$

Because $\mathrm{pred}(H(P)) \cap \mathrm{pred}(B(P)^-) \subseteq E$ and $\mathrm{pred}(B(P')^-) \cap E = \emptyset$, all rules $r \in G'$ satisfy $B(r)^- \cap J' = \emptyset$ and we obtain

$$\begin{aligned} S_{G'}^{I^c}(J^c) &= S_{G'}^{I^c}(J^c \cup J') \\ &\subseteq S_G^{I^c}(J^c \cup E^c \cup J'). \end{aligned}$$

To show $J' \subseteq J$, we use $I \subseteq I'$ and Proposition 20 (a):

$$S_G^{J^c}(I^c \cup I') \subseteq S_G^{J^c}(I^c \cup I). \qquad \square$$

Theorem 41. *Let $(P_i)_{i\in\mathbb{I}}$ be an instantiation sequence for aggregate program P and $(P_j)_{j\in\mathbb{J}}$ be a refinement of $(P_i)_{i\in\mathbb{I}}$.*

Then, $AM((P_i)_{i\in\mathbb{I}}) \leq_p AM((P_j)_{j\in\mathbb{J}}) \leq_p WM(\tau^*(P))$.

Proof of Theorem 41. We first show $AM((P_j)_{j\in\mathbb{J}}) \leq_p WM(\tau^*(P))$ and then $AM((P_i)_{i\in\mathbb{I}}) \leq_p AM((P_j)_{j\in\mathbb{J}})$.

Property ($AM((P_j)_{j\in\mathbb{J}}) \leq_p WM(\tau^*(P))$). Let E_j, (I_j^c, J_j^c), and (I_j, J_j) be defined as in (14) to (16) for the sequence $(P_j)_{j\in\mathbb{J}}$. Similarly, let E_j', $(I_j^{c\prime}, J_j^{c\prime})$, and (I_j', J_j') be defined as in (3) to (5) for the sequence $(G_j)_{j\in\mathbb{J}}$ with $G_j = \tau^*(P_j)$. Furthermore, let E_j^c be the set of all ground atoms over atoms in E_j.

We first show $E_j' \subseteq E_j^c$ for each $j \in \mathbb{J}$ by showing that $E_j' \subseteq E_j^c$. By Lemma 39, only negative body literals have to be taken into account:

$$\begin{aligned} E_j' &= B(G_j)^{\pm} \cap \bigcup_{j<k} H(G_k) \\ &= B(G_j)^- \cap \bigcup_{j<k} H(G_k). \end{aligned}$$

Observe that $\mathrm{pred}(B(G_j)^- \subseteq \mathrm{pred}(B(P_j)^-))$ and $\mathrm{pred}(H(G_j)) \subseteq \mathrm{pred}(H(P_j))$. Thus, we get

$$\begin{aligned}\mathrm{pred}(E'_j) &= \mathrm{pred}(B(G_j)^- \cap \bigcup_{j<k} H(G_k)) \\ &\subseteq \mathrm{pred}(B(P_j)^-) \cap \mathrm{pred}(\bigcup_{j<k} H(P_k)) \\ &\subseteq \mathrm{pred}(B(P_j)^-) \cap \mathrm{pred}(\bigcup_{j\leq k} H(P_k)) \\ &= E_j.\end{aligned}$$

It follows that $E'_j \subseteq E^{\mathrm{c}}_j$.

By Theorem 25, we have $\bigsqcup_{j\in\mathbb{J}}(I'_j, J'_j) \leq_p WM(\tau^*(G))$. To show the theorem, we show $(I_j, J_j) \leq_p (I'_j, J'_j)$. We omit the full induction proof and focus on the key idea: Using Lemma 40, whose precondition holds by construction of E_j, and Proposition 22, we get

$$\begin{aligned}AM^{I^{\mathrm{c}}_j, J^{\mathrm{c}}_j}_{E_j}(P_j) &\leq_p WM^{(I^{\mathrm{c}}_j, J^{\mathrm{c}}_j)\sqcup(\emptyset, E^{\mathrm{c}}_j)}(G_j) \\ &\leq_p WM^{(I^{\mathrm{c}\prime}_j, J^{\mathrm{c}\prime}_j)\sqcup(\emptyset, E'_j)}(G_j).\end{aligned}$$

Property $(AM((P_i)_{i\in\mathbb{I}}) \leq_p AM((P_{i,j})_{(i,j)\in\mathbb{J}}))$. We omit a full induction proof for this property because it would be very technical. Instead, we focus on the key idea why the approximate model of a refined instantiation sequence is at least as precise as the one of an instantiation sequence.

Let E_i and $E_{i,j}$ be defined as in (14) for the instantiation and refined instantiation sequence, respectively. Clearly, we have $E_{i,j} \subseteq E_i$ for each $(i, j) \in \mathbb{J}$. Observe, that (due to rule dependencies and Observation 21 (c)) calculating the approximate model of the refined sequence, using E_i instead of $E_{i,j}$ in (16), would result in the same approximate model as for the instantiation sequence. With this, the property simply follows from the monotonicity of the stable operator. □

Theorem 42. *Let $(P_i)_{i\in\mathbb{I}}$ be an instantiation sequence of an aggregate program P such that $E_i = \emptyset$ for each $i \in \mathbb{I}$ as defined in* (14)*.*

Then, $AM((P_i)_{i\in\mathbb{I}})$ is total.

Proof of Theorem 42. Clearly, we have $E_i = \emptyset$ if all components are stratified. With this, the theorem follows from Lemma 38. □

Theorem 43. *Let $(P_i)_{i\in\mathbb{I}}$ be a (refined) instantiation sequence of an aggregate program P, and let $(I^{\mathrm{c}}_i, J^{\mathrm{c}}_i)$ and (I_i, J_i) defined as in* (15) *and* (16)*.*

Then, $\bigcup_{i\in\mathbb{I}} \tau^*_{J^{\mathrm{c}}_i \cup J_i}(P_i)^{(I^{\mathrm{c}}_i, J^{\mathrm{c}}_i)\sqcup(I_i, J_i)}$ and $\tau^*(P)$ have the same well-founded and stable models.

Proof of Theorem 43. Let E_i, $(I^{\mathrm{c}}_i, J^{\mathrm{c}}_i)$, and (I_i, J_i) be defined as in (14) to (16) for the sequence $(P_i)_{i\in\mathbb{I}}$. Similarly, let E'_i, $(I^{\mathrm{c}\prime}_i, J^{\mathrm{c}\prime}_i)$, and (I'_i, J'_i) be defined

as in (3) to (5) for the sequence $(G_i)_{i\in\mathbb{I}}$ with $G_i = \tau^*(P_i)$. Furthermore, we assume w.l.o.g. that $\mathbb{I} = \{1,\dots,n\}$.

We have already seen in the proof of Theorem 41, that the atoms E_i' are a subset of the ground atoms over predicates E_i and that $(I_i, J_i) \leq_p (I_i', J_i')$. Observing that ground atoms over predicates E_i can only appear negatively in rule bodies, we obtain that $G_i^{(I_i^{\mathrm{c}\prime}, J_i^{\mathrm{c}\prime})\cup(I_i', J_i')\cup(\emptyset, E_i')} = G_i^{(I_i^{\mathrm{c}\prime}, J_i^{\mathrm{c}\prime})\cup(I_i', J_i')}$. By Theorem 28 and Corollary 30, we obtain that $\bigcup_{i\in\mathbb{I}} G_i^{(I_i^{\mathrm{c}}, J_i^{\mathrm{c}})\sqcup(I_i, J_i)}$ and $\tau^*(P)$ have the same well-founded and stable models. To shorten the notation, we let

$$F_i = \tau^*(P_i)^{(I_i^{\mathrm{c}}, J_i^{\mathrm{c}})\sqcup(I_i, J_i)}, \qquad H_i = \tau^*_{J_i^{\mathrm{c}}\cup J_i}(P_i)^{(I_i^{\mathrm{c}}, J_i^{\mathrm{c}})\sqcup(I_i, J_i)},$$
$$F = \bigcup_{i\in\mathbb{I}} F_i, \text{ and} \qquad H = \bigcup_{i\in\mathbb{I}} H_i.$$

With this, it remains to show that programs F and H have the same well-founded and stable models.

We let $J = \bigcup_{i\in\mathbb{I}} J_i$. Furthermore, we let $\tau^*(a)$ be a subformula in F_i and $\tau^*_{J_i^{\mathrm{c}}\cup J_i}(a)$ be a subformula in H_i where both subformulas originate from the translation of the closed aggregate a. (We see below that existence of one implies the existence of the other because both formulas are identical in their context.)

Because an aggregate always depends positively on the predicates occurring in its elements, the intersection between $\bigcup_{i<k} H(F_k) = \bigcup_{i<k} J_k$ and the atoms occurring in $\tau^*(a)$ is empty. Thus the two formulas $\tau^*_{J_i^{\mathrm{c}}\cup J_i}(a)$ and $\tau^*_J(a)$ are identical. Observe that each stable model of either F and H is a subset of J. By Proposition 35, satisfiability of the aggregates formulas as well as their reducts is the same for subsets of J. Thus, both formulas have the same stable models. Similarly, the well-founded model of both formulas must be more-precise than $(\emptyset, J)$. By Proposition 35, satisfiability of the aggregate formulas as well as their ID-reducts is the same. Thus, both formulas have the same well-founded model. □

Proposition 44. *Let r be a safe normal rule, (I, J) be a finite four-valued interpretation, $f \in \{\mathbf{t}, \mathbf{f}\}$, and J' be a finite two-valued interpretation.*

Then, a call to $\texttt{GroundRule}^{I,J}_{r,f,J'}(\iota, B(r))$ *returns the finite set of instances g of r satisfying*

$$J \models \tau(B(g))^{\wedge}_I \text{ and } (f = \mathbf{t} \text{ or } B(g)^+ \not\subseteq J'). \tag{17}$$

Proof of Proposition 44. Observe that the algorithm does not modify f, r, (I, J), and J'. To shorten the notation below, let $G_{\sigma,L} = \texttt{GroundRule}^{I,J}_{r,f,J'}(\sigma, L)$.

Calling $G_{\iota,B(r)}$, the algorithm maintains the following invariants in subsequent calls $G_{\sigma,L}$:

(1) $(B(r) \setminus L)\sigma^+ \subseteq J$,

(2) $(B(r) \setminus L)\sigma^- \cap I = \emptyset$, and

(3) each comparison in $(B(r) \setminus L)\sigma$ holds.

We only prove the first invariant because the latter two can be shown in a similar way. We prove by induction.

Base. For the call $G_{\iota,B(r)}$, the invariant holds because the set difference $B(r) \setminus L$ is empty for $L = B(r)$.

Hypothesis. We assume the invariant holds for call $G_{\sigma,L}$ and show that it is maintained in subsequent calls.

Step. Observe that there are only further calls if L is non-empty. In Line 3, a body literal l is selected from L. Observe that it is always possible to select such a literal. In case that there are positive literals in L, we can simply select one of them. In case that there are no positive literals in L, σ replaces all variables in the positive body of r. Because r is safe, all literals in $L\sigma$ are ground and we can select any one of them.

In the case that l is a positive literal, all substitutions σ', obtained by calling $\texttt{Matches}_l^{I,J}(\sigma)$ in the following line, ensure

$$l\sigma' \in J.$$

Furthermore, σ is more general than σ'. Thus, we have

$$\begin{aligned}(B(r) \setminus L)\sigma'^+ &= (B(r) \setminus L)\sigma^+ \\ &\subseteq J.\end{aligned}$$

In Line 5, the algorithm calls $G_{\sigma',L'}$ with $L' = L \setminus \{l\}$. We obtain

$$\begin{aligned}(B(r) \setminus L')\sigma'^+ &= (B(r) \setminus L)\sigma'^+ \cup \{l\sigma'\} \\ &\subseteq J.\end{aligned}$$

In the case that l is a negative literal or a comparison, we get $(B(r) \setminus L)^+ = (B(r) \setminus L \setminus \{l\})^+$. Furthermore, the substitution σ is either not changed or is discarded altogether. Thus, the invariant is maintained in subsequent calls to `GroundRule`.

We prove by induction over subsets L of $B(r)$ with corresponding substitution σ satisfying invariants (1)–(3) that $G_{L,\sigma}$ is finite and that $g \in G_{L,\sigma}$ iff g is a ground instance of $r\sigma$ that satisfies (17).

Base. We show the base case for $L = \emptyset$. Using invariant (1), we only have to consider substitutions σ with $B(r)^+\sigma \subseteq J$. Because r is safe and σ replaces all variables in its positive body, σ also replaces all variables in its head and negative body. Thus, $r\sigma$ is ground and the remainder of the algorithm just

filters the set $\{r\sigma\}$ while the invariants (1)–(3) ensure that $J \models \tau(B(r\sigma))^{\wedge}_{I}$. The condition in Line 2 cannot apply because $L = \emptyset$. The condition in Line 9 discards rules $r\sigma$ not satisfying $f = \mathbf{t}$ or $B(r\sigma)^{+} \not\subseteq J'$.

Hypothesis. We show that the property holds for $L \neq \emptyset$ assuming that it holds for subsets $L' \subset L$ with corresponding substitutions σ'.

Step. Because $L \neq \emptyset$ we only have to consider the case in Line 2.

First, the algorithm selects an element $l \in L$. We have already seen that it is always possible to select such an element. Let $L' = L \setminus \{l\}$. The algorithm then loops over the set

$$\Sigma = \texttt{Matches}_{l}^{I,J}(\sigma)$$

and, in Lines 4 to 5, computes the union

$$G_{\sigma,L} = \bigcup_{\sigma' \in \Sigma} G_{\sigma',L'}.$$

First, we show that the set $G_{\sigma,L}$ is finite. In case l is not a positive literal, the set Σ has at most one element. In case l is a positive literal, observe that there is a one-to-one correspondence between Σ and the set $\{l\sigma' \mid \sigma' \in \Sigma\}$. We obtain that Σ is finite because $\{l\sigma' \mid \sigma' \in \Sigma\} \subseteq J$ and J is finite. Furthermore, using the induction hypothesis, each set $G_{\sigma',L'}$ in the union $G_{\sigma,L}$ is finite. Hence, the set $G_{\sigma,L}$ returned by the algorithm is finite.

Second, we show $g \in G_{\sigma,L}$ implies that g is a ground instance of $r\sigma$ satisfying (17). We have that g is a member of some $G_{\sigma',L'}$. By the induction hypothesis, g is a ground instance of $r\sigma'$ satisfying (17). Observe that g is also a ground instance of $r\sigma$ because σ is more general than σ'.

Third, we show that each ground instance g of $r\sigma$ satisfying (17) is also contained in $G_{\sigma,L}$. Because g is a ground instance of $r\sigma$, there is a substitution θ more specific than σ such that $g = r\theta$. In case that the selected literal $l \in L$ is a positive literal, we have $l\theta \in J$. Then, there is also a substitution θ' such that $\theta' \in \texttt{match}(l\sigma, l\theta)$. Let $\sigma' = \sigma \circ \theta'$. By Definition 16, we have $\sigma' \in \Sigma$. It follows that $g \in G_{\sigma,L}$ because $g \in G_{\sigma',L'}$ by the induction hypothesis and $G_{\sigma',L'} \subseteq G_{\sigma,L}$. In the case that l is not a positive literal, we have $\sigma \in \Sigma$ and can apply a similar argument.

Hence, we have shown that the proposition holds for $G_{\iota,B(r)}$. □

Corollary 45. *Let r be a safe normal rule and (I, J) be a finite four-valued interpretation.*

Then, $\mathrm{Inst}^{I,J}(\{r\}) = \texttt{GroundRule}_{r,\mathbf{t},\emptyset}^{I,J}(\iota, B(r))$.

Proof of Corollary 45. The corollary directly follows from Proposition 44 and the definition of $\mathrm{Inst}^{I,J}(\{r\})$. □

Lemma 46. *Let r be a safe normal rule, (I, J) be a finite four-valued interpretation, and $J' \subseteq J$ be a two-valued interpretation.*

Then, we have

$$\mathrm{Inst}^{I,J}(\{r\}) = \mathrm{Inst}^{I,J'}(\{r\}) \cup \mathtt{GroundRule}^{I,J}_{r,\mathbf{f},J'}(\iota, B(r)).$$

Proof of Lemma 46. Let G be the set of all ground instances of r and

$$G^{X,Y}_f = \{g \in G \mid Y \models \tau(B(g))^\wedge_I, (f = \mathbf{t} \text{ or } B(g)^+ \not\subseteq X)\}.$$

By Proposition 44 and Corollary 45, we can reformulate the lemma as $G^{\emptyset,J}_{\mathbf{t}} = G^{\emptyset,J'}_{\mathbf{t}} \cup G^{J',J}_{\mathbf{f}}$. We have

$$\begin{aligned}
G^{\emptyset,J}_{\mathbf{t}} &= \{g \in G \mid J \models \tau(B(g))^\wedge_I\} \\
G^{\emptyset,J'}_{\mathbf{t}} &= \{g \in G \mid J' \models \tau(B(g))^\wedge_I\}, \text{ and} \\
G^{J',J}_{\mathbf{f}} &= \{g \in G \mid J \models \tau(B(g))^\wedge_I, B(g)^+ \not\subseteq J'\}.
\end{aligned}$$

Observe that, given $J' \subseteq J$, we can equivalently write $G^{\emptyset,J'}_{\mathbf{t}}$ as

$$G^{\emptyset,J'}_{\mathbf{t}} = \{g \in G \mid J \models \tau(B(g))^\wedge_I, B(g)^+ \subseteq J'\}.$$

Because $B(g)^+ \subseteq J'$ and $B(g)^+ \not\subseteq J'$ cancel each other, we get

$$G_{\emptyset,J} = G_{\emptyset,J'} \cup G_{J',J}. \qquad \square$$

Proposition 47. *Let P be an aggregate program, (I, J) be a finite four-valued interpretation, $G^\epsilon = \mathrm{Inst}^{I,J}(P^\epsilon)$, $G^\eta = \mathrm{Inst}^{I,J}(P^\eta)$, $J^\alpha = \mathtt{Propagate}^{I,J}_P(G^\epsilon, G^\eta)$, and $G^\alpha = \mathrm{Inst}^{I,J\cup J^\alpha}(P^\alpha)$.*

Then,

(a) $\mathtt{Assemble}(G^\alpha, G^\eta) = \tau^*_J(P)^{I,J}$ *and*
(b) $H(G^\alpha) = T_{\tau^*(P)_I}(J)$.

Proof of Proposition 47. We first show Property (a) and then (b).

Property (a). For a rule $r \in P$, we use r^α to refer to the corresponding rule with replaced aggregate occurrences in P^α. Similarly, for a ground instance g of r, we use g^α to to refer to the corresponding instance of r^α. Observe that $\tau^*_J(P)^{I,J} = \tau^*_J(\mathrm{Inst}^{I,J}(P))$. We show that $g \in \mathrm{Inst}^{I,J}(P)$ iff $g^\alpha \in \mathrm{Inst}^{I,J\cup J^\alpha}(P^\alpha)$. In the following, because the rule bodies of g and g^α only differ regarding aggregates and their replacement atoms, we only consider rules with aggregates in their bodies.

Case $g \in \mathrm{Inst}^{I,J}(P)$. Let r be a rule in P containing aggregate a, α be the replacement atom of form (21) for a, and σ be a ground substitution such that $r\sigma = g$. We show that for each aggregate $a\sigma \in B(g)$, we have $\epsilon_{r,a}(G^\epsilon, \sigma) \cup G \neq \emptyset$ and $J \models \tau^*_G(a\sigma)_I$ with $G = \eta_{r,a}(G^\eta, \sigma)$ and in turn $J^\alpha \models \alpha\sigma$. Because $J \models \tau^*(B(g))^\wedge_I$, we get $J \models \tau^*_J(a\sigma)_I$. It remains to show that $\epsilon_{r,a}(G^\epsilon, \sigma) \cup G \neq \emptyset$ and $\tau^*_J(a\sigma) = \tau^*_G(a\sigma)$. Observe that $\tau^*_J(a\sigma) = \tau^*_G(a\sigma)$ because the set G obtained from rules in G^η contains all instances of elements of $a\sigma$ whose conditions are satisfied by J while the remaining literals of these

rules are contained in the body of g. Furthermore, observe that if no aggregate element is satisfied by J, we get $\epsilon_{r,a}(G^\epsilon, \sigma) \neq \emptyset$ because the corresponding ground instance of (22) is satisfied.

Case $g^\alpha \in \mathrm{Inst}^{I,J\cup J^\alpha}(P^\alpha)$. Let r^α be a rule in P^α containing replacement atom α of form (21) for aggregate a and σ be a ground substitution such that $r^\alpha\sigma = g^\alpha$. Because $J \cup J^\alpha \models \tau(B(g^\alpha))^\wedge_I$, we have $\alpha\sigma \in J^\alpha$. Thus, we get that $J \models \tau^*_G(a\sigma)_I$ with $G = \eta_{r,a}(G^\eta, \sigma)$. We have already seen in the previous case that $\tau^*_J(a\sigma) = \tau^*_G(a\sigma)$. Thus, $J \models \tau^*_J(a\sigma)_I$. Observing that $g = r\sigma$ and $a\sigma \in B(g)$, we get $g \in \mathrm{Inst}^{I,J}(P)$.

Property (b). This property follows from Property (a), Proposition 35, and Lemma 12. □

Proposition 48. *Let P be an aggregate program, (I^c, J^c) be a finite four-valued interpretation, and $J = S^{J^c}_{\tau^*(P)}(I^c)$.*

Then,

(a) $\texttt{GroundComponent}(P, I^c, J^c)$ *terminates iff J is finite.*

If J is finite, then

(b) $\texttt{GroundComponent}(P, I^c, J^c) = \tau^*_{J^c\cup J}(P)^{I^c, J^c\cup J}$ *and*

(c) $H(\texttt{GroundComponent}(P, I^c, J^c)) = J$.

Proof of Proposition 48. We prove Properties (a) and (b) by showing that the function calculates the stable model by iteratively calling the T operator until a fixed-point is reached.

Property (a) *and* (b). At each iteration i of the loop starting with 1, let J^α_i be the value of $\texttt{Propagate}^{I,J}_P(G^\epsilon, G^\eta)$ in Line 6, G^ϵ_i, G^η_i, and G^α_i be the values on the right-hand-side of the assignments in Lines 4, 5 and 7, and $J_i = H(G^\alpha_i)$. Furthermore, let $J_0 = \emptyset$.

By Corollary 45 and Lemma 46, we get

$$\begin{aligned}
G^\epsilon_i &= \mathrm{Inst}^{I^c, J^c\cup J_{i-1}}(P^\epsilon),\\
G^\eta_i &= \mathrm{Inst}^{I^c, J^c\cup J_{i-1}}(P^\eta),\\
J^\alpha_i &= \texttt{Propagate}^{I,J}_P(G^\epsilon_i, G^\eta_i), \text{ and}\\
G^\alpha_i &= \mathrm{Inst}^{I^c, J^c\cup J^\alpha_i\cup J_{i-1}}(P^\alpha).
\end{aligned}$$

Using Proposition 47 (b), and observing the one-to-one correspondence between G^α_i and $\tau^*(P)^{I^c, J^c\cup J^c_i}$ we get

$$\begin{aligned}
J_i &= H(G^\alpha_i)\\
&= T_{\tau^*(P)_{I^c}}(J^c \cup J_{i-1}).
\end{aligned}$$

Observe that, if the loop exits, then the algorithm computes the fixed point of $T^{J^c}_{\tau^*(P)_{I^c}}$, i.e., $J = S^{I^c}_{\tau^*(P)}(J^c)$. Furthermore, observe that this fixed point calculation terminates whenever $S^{I^c}_{\tau^*(P)}(J^c)$ is finite. Finally, we obtain $\texttt{GroundComponent}(P, I^c, J^c) = \tau^*_{J^c\cup J}(P)^{I^c, J^c\cup J}$ using Proposition 47 (a).

Property (c). We have seen above that J is a fixed point of $T^{J^c}_{\tau^*(P)_{I^c}}$. By Proposition 47 (b) and observing that function `Assemble` only modifies rule bodies, we get $H(\texttt{GroundComponent}(P, I^c, J^c)) = J$. □

Theorem 49. *Let P be an aggregate program, $(P_i)_{i\in\mathbb{I}}$ be a refined instantiation sequence for P, and E_i, (I_i^c, J_i^c), and (I_i, J_i) be defined as in* (14) *to* (16).

If $(P_i)_{i\in\mathbb{I}}$ is selected by Algorithm 3 in Line 2, then we have that

(a) *the call* `Ground`(P) *terminates iff $AM((P_i)_{i\in\mathbb{I}})$ is finite, and*

(b) *if $AM((P_i)_{i\in\mathbb{I}})$ is finite, then* `Ground`$(P) = \bigcup_{i\in\mathbb{I}} \tau^*_{J_i^c\cup J_i}(P_i)^{(I_i^c,J_i^c)\sqcup(I_i,J_i)}$.

Proof of Theorem 49. Since the program is finite, its instantiation sequences are finite, too. We assume w.l.o.g. that $\mathbb{I} = \{1, \ldots, n\}$ for some $n \geq 0$. We let F_i^c and G_i^c be the values of variables F and G at iteration i at the beginning of the loop in Lines 4 to 7, and F_i and G_i be the results of the calls to `GroundComponent` in Lines 6 and 7 at iteration i.

By Proposition 48, we get that Lines 5 to 7 correspond to an application of the approximate model operator as Definition 14. For each iteration i, we get

$$\begin{aligned}(F_i^c, G_i^c) &= \bigsqcup_{j<i}(F_i, G_i),\\ (I_i^c, J_i^c) &= (H(F_i^c), H(G_i^c)),\\ (I_i, J_i) &= (H(F_i), H(G_i)), \text{ and}\\ G_i &= \tau^*_{J_i^c\cup J_i}(P_i)^{(I_i^c,J_i^c)\sqcup(I_i,J_i)}\end{aligned}$$

whenever (I_i, J_i) is finite. In the case that each (I_i, J_i) is finite, the algorithm returns in Line 8 the program

$$\begin{aligned}G_n^c \cup G_n &= \bigcup_{i\in\mathbb{I}} G_i\\ &= \bigcup_{i\in\mathbb{I}} \tau^*_{J_i^c\cup J_i}(P_i)^{(I_i^c,J_i^c)\sqcup(I_i,J_i)}.\end{aligned}$$

Clearly, the algorithm terminates iff each call to `GroundComponent` is finite, which is exactly the case when $AM((P_i)_{i\in\mathbb{I}})$ is finite. □

Corollary 50. *Let P be an aggregate program.*

If `Ground`(P) *terminates then P and* `Ground`(P) *have the same well-founded and stable models.*

Proof of Corollary 50. This is a direct consequence of Theorems 43 and 49. □

Proposition 51. *Let a be a closed aggregate.*

If a is antimonotone, then $\tau^*(a)_I$ is classically equivalent to $\top$ if $I \models \tau^*(a)$ and $\bot$ otherwise for any two-valued interpretation I.

Proof of Proposition 51. Let G be the set of ground instances of the aggregate elements of a and $D \subseteq G$ be a set such that $D \not\models a$.

Due to the antimonotonicity of the aggregate, we get $\tau_a^*(D) = \bot$. Thus, the reduct is constant because all consequents in $\tau^*(a)$ as well as $\tau^*(a)_I$ are equal to $\bot$ and the antecedents in $\tau^*(a)$ are completely evaluated by the reduct. Hence, the lemma follows by Proposition 6 (b). □

Proposition 52. *Let I be a finite two-valued interpretation, E be a set of aggregate elements, and b be an integer.*

For $T = H(\{e \in \mathrm{Inst}(E) \mid I \models B(e)\})$, we get

(a) *$\tau^*(\#\mathrm{sum}\{E\} \succ b)_I$ is classically equivalent to $\tau^*(\#\mathrm{sum}^+\{E\} \succ b')$ with $\succ \in \{\geq, >\}$ and $b' = b - \#\mathrm{sum}^-(T)$, and*

(b) *$\tau^*(\#\mathrm{sum}\{E\} \prec b)_I$ is classically equivalent to $\tau^*(\#\mathrm{sum}^-\{E\} \prec b')$ with $\prec \in \{\leq, <\}$ and $b' = b - \#\mathrm{sum}^+(T)$.*

Proof of Proposition 52. We only show Property (a) because the proof of Property (b) is symmetric.

Let $a = \#\mathrm{sum}\{E\} \succ b$ and $a_+ = \#\mathrm{sum}^+\{E\} \succ b'$. Given an arbitrary two-valued interpretation J, we consider the following two cases:

Case $J \not\models \tau^*(a)_I$. There is a set $D \subseteq G$ such that $D \not\models a$, $I \models \tau(D)^\wedge$, and $J \not\models \tau_a^*(D)^\vee$. Let $\widehat{D} = D \cup \{e \in G \mid I \models \tau(e), w(H(e)) < 0\}$.

Clearly, $\widehat{D} \not\models a$ and $I \models \tau(\widehat{D})^\wedge$. Furthermore, $J \not\models \tau_a^*(\widehat{D})^\vee$ because we constructed $\widehat{D}$ so that $\tau_a^*(\widehat{D})^\vee$ is stronger than $\tau_a^*(D)^\vee$ because more elements with negative weights have to be taken into considerations.

Next, observe that $\widehat{D} \not\models a_+$ holds because we have $\#\mathrm{sum}^+(H(\widehat{D})) = \#\mathrm{sum}^+(H(D))$ and $\#\mathrm{sum}^-(H(\widehat{D})) = \#\mathrm{sum}^-(T)$, which corresponds to the value subtracted from the bound of a_+. To show that $J \not\models \tau_{a_+}^*(\widehat{D})^\vee$, we show $\tau_{a_+}^*(\widehat{D})^\vee$ is stronger than $\tau_a^*(\widehat{D})^\vee$. Let $C \subseteq G \setminus \widehat{D}$ be a set of elements such that $\widehat{D} \cup C \rhd a_+$. Because the justification of a_+ is independent of elements with negative weights, each clause in $\tau_{a_+}^*(\widehat{D})^\vee$ involving an element with a negative weight is subsumed by another clause without that element. Thus, we only consider sets C containing elements with positive weights. Observe that $\widehat{D} \cup C \rhd a$ holds because we have $\#\mathrm{sum}(H(\widehat{D} \cup C)) = \#\mathrm{sum}^+(H(\widehat{D} \cup C)) + \#\mathrm{sum}^-(T)$. Hence, we get $J \not\models \tau_{a_+}^*(\widehat{D})^\vee$.

Case $J \not\models \tau^*(a_+)_I$. There is a set $D \subseteq G$ such that $D \not\models a_+$, $I \models \tau(D)^\wedge$, and $J \not\models \tau_{a_+}^*(D)^\vee$. Let $\widehat{D} = D \cup \{e \in G \mid I \models \tau(e), w(H(e)) < 0\}$.

Observe that $\widehat{D} \not\models a_+$, $I \models \tau(\widehat{D})^\wedge$, and $J \not\models \tau_{a_+}^*(\widehat{D})^\vee$. As in the previous case, we can show that $\tau_a^*(\widehat{D})^\vee$ is stronger than $\tau_{a_+}^*(\widehat{D})^\vee$ because clauses in $\tau_a^*(\widehat{D})^\vee$ involving elements with negative weights are subsumed. Hence, we get $J \not\models \tau_a^*(\widehat{D})^\vee$. □

Proposition 53. *Let I be a two-valued interpretation, E be a set of aggregate elements, and b be a ground term.*

We get the following properties:

(a) $\tau^*(f\{E\} < b)_I \vee \tau^*(f\{E\} > b)_I$ *implies* $\tau^*(f\{E\} \neq b)_I$, *and*
(b) $\tau^*(f\{E\} = b)_I$ *implies* $\tau^*(f\{E\} \leq b)_I \wedge \tau^*(f\{E\} \geq b)_I$.

Proof of Proposition 53. Let G be the set of ground instances of E, $a_\prec = f\{E\} \prec b$ for aggregate relation $\prec$, and J be a two-valued interpretation.

Property (a). We show that $J \models \tau^*(a_<)_I \vee \tau^*(a_>)_I$ implies $J \models \tau^*(a_\neq)_I$.

Case $J \models \tau^*(a_<)_I$. Observe that $\tau^*(a_\neq)$ is conjunction of implications of form $\tau(D)^\wedge \rightarrow \tau^*_{a_\neq}(D)^\vee$ with $D \subseteq G$ and $D \not\models a_\neq$. Furthermore, note that $D \not\models a_\neq$ implies $D \not\models a_<$. Thus, $\tau^*(a_<)$ contains the implication $\tau(D)^\wedge \rightarrow \tau^*_{a_<}(D)^\vee$. Because $J \models \tau^*(a_<)_I$, we get $I \not\models \tau(D)^\wedge$ or $J \models \tau^*_{a_<}(D)^\vee$. Hence, the property holds in this case because $J \models \tau^*_{a_<}(D)^\vee$ implies $J \models \tau^*_{a_\neq}(D)^\vee$.

Case $J \models \tau^*(a_>)_I$. The property can be shown analogously for this case.

Property (a). This property can be shown in a similar way as the previous one. We show by contraposition that $J \models \tau^*(a_=)_I$ implies $J \models \tau^*(a_\leq)_I \wedge \tau^*(a_\geq)_I$.

Case $J \not\models \tau^*(a_\leq)_I$. Observe that $\tau^*(a_\leq)$ is conjunction of implications of form $\tau(D)^\wedge \rightarrow \tau^*_{a_\leq}(D)^\vee$ with $D \subseteq G$ and $D \not\models a_\leq$. Furthermore, note that $D \not\models a_\leq$ implies $D \not\models a_=$. Thus, $\tau^*(a_=)$ contains the implication $\tau(D)^\wedge \rightarrow \tau^*_{a_=}(D)^\vee$. Because $J \not\models \tau^*(a_\leq)_I$, we get $I \models \tau(D)^\wedge$ and $J \not\models \tau^*_{a_<}(D)^\vee$ for some $D \subseteq G$ with $D \not\models a_\leq$. Hence, the property holds in this case because $J \not\models \tau^*_{a_\leq}(D)^\vee$ implies $J \not\models \tau^*_{a_=}(D)^\vee$.

Case $J \not\models \tau^*(a_\geq)_I$. The property can be shown analogously for this case. □

Proposition 54. *Let I and J be two-valued interpretations, f be an aggregate function among* #count, #sum$^+$, #sum$^-$ *or* #sum, *E be a set of aggregate elements, and b be an integer.*

We get the following properties:

(a) *for $I \subseteq J$, we have $J \models \tau^*(f\{E\} < b)_I \vee \tau^*(f\{E\} > b)_I$ iff $J \models \tau^*(f\{E\} \neq b)_I$, and*
(b) *for $J \subseteq I$, we have $J \models \tau^*(f\{E\} = b)_I$ iff $J \models \tau^*(f\{E\} \leq b)_I \wedge \tau^*(f\{E\} \geq b)_I$.*

Proof of Proposition 54. We only consider the case that f is the #sum function because the other ones are special cases of this function. Furthermore, we only consider the only if directions because we have already established the other directions in Proposition 53.

Let G be the set of ground instances of E, $T_I = H(\{g \in G \mid I \models B(g)\})$, and $T_J = H(\{g \in G \mid J \models B(g)\})$.

Property (a). Because $I \subseteq J$, we get $\#\text{sum}^+(T_I) \leq \#\text{sum}^+(T_J)$ and $\#\text{sum}^-(T_J) \leq \#\text{sum}^-(T_I)$. We prove by contraposition.

Case $J \not\models \tau^*(a_<)_I$ *and* $J \not\models \tau^*(a_>)_I$. We use Propositions 34 and 52 to get the following two inequalities:

$$\begin{aligned}\#\mathrm{sum}^-(T_J) &\geq b - \#\mathrm{sum}^+(T_I) \text{ because } J \not\models \tau^*(a_<)_I \text{ and}\\ \#\mathrm{sum}^+(T_J) &\leq b - \#\mathrm{sum}^-(T_I) \text{ because } J \not\models \tau^*(a_>)_I.\end{aligned}$$

Using $\#\mathrm{sum}^-(T_J) \leq \#\mathrm{sum}^-(T_I)$, we can rearrange as

$$\begin{aligned}b - \#\mathrm{sum}^+(T_I) &\leq \#\mathrm{sum}^-(T_J)\\ &\leq \#\mathrm{sum}^-(T_I)\\ &\leq b - \#\mathrm{sum}^+(T_J).\end{aligned}$$

Using $\#\mathrm{sum}^+(T_I) \leq \#\mathrm{sum}^+(T_J)$, we obtain

$$\#\mathrm{sum}^+(T_I) = \#\mathrm{sum}^+(T_J).$$

Using $\#\mathrm{sum}^+(T_I) = \#\mathrm{sum}^+(T_J)$, we get

$$\begin{aligned}b - \#\mathrm{sum}^+(T_I) &\leq \#\mathrm{sum}^-(T_J)\\ &\leq \#\mathrm{sum}^-(T_I)\\ &\leq b - \#\mathrm{sum}^+(T_I)\end{aligned}$$

and, thus, obtain

$$\begin{aligned}\#\mathrm{sum}^-(T_I) &= \#\mathrm{sum}^-(T_J) \text{ and}\\ b &= \#\mathrm{sum}(T_I)\\ &= \#\mathrm{sum}(T_J).\end{aligned}$$

Observe that this gives rise to an implication in $\tau^*(a_\neq)_I$ that is not satisfied by J. Hence, we get $J \not\models \tau^*(a_\neq)_I$.

Property (b). Because $J \subseteq I$, we get $\#\mathrm{sum}^+(T_J) \leq \#\mathrm{sum}^+(T_I)$ and $\#\mathrm{sum}^-(T_I) \leq \#\mathrm{sum}^-(T_J)$.

Case $J \models \tau^*(a_\leq)_I$ *and* $J \models \tau^*(a_\geq)_I$. Using Propositions 34 and 52, we get

$$\begin{aligned}\#\mathrm{sum}^+(T_J) &\geq b - \#\mathrm{sum}^-(T_I) \text{ because } J \models \tau^*(a_\geq)_I \text{ and}\\ \#\mathrm{sum}^-(T_J) &\leq b - \#\mathrm{sum}^+(T_I) \text{ because } J \models \tau^*(a_\leq)_I.\end{aligned}$$

Observe that we can proceed as in the proof of the previous property because the relation symbols are just flipped. We obtain

$$\begin{aligned}\#\mathrm{sum}^-(T_I) &= \#\mathrm{sum}^-(T_J) \text{ and}\\ b &= \#\mathrm{sum}(T_I)\\ &= \#\mathrm{sum}(T_J).\end{aligned}$$

We get $J \models \tau^*(a_=)_I$ because for any subset of tuples in T_I that do not satisfy the aggregate, we have additional tuples in T_J that satisfy the aggregate. □

Proposition 55. *Let I and J be finite two-valued interpretations, f be an aggregate function, E be a set of aggregate elements, and b be a ground term.*

For $T_I = \{H(e) \mid e \in \mathrm{Inst}(E), I \models B(e)\}$ and $T_J = \{H(e) \mid e \in \mathrm{Inst}(E), J \models B(e)\}$, we get the following properties:

(a) *for $J \subseteq I$, we have $J \models \tau^*(f\{E\} \neq b)_I$ iff there is no set $X \subseteq T_I$ such that $f(X \cup T_J) = b$, and*
(b) *for $I \subseteq J$, we have $J \models \tau^*(f\{E\} = b)_I$ and iff there is a set $X \subseteq T_J$ such that $f(X \cup T_I) = b$.*

Proof of Proposition 55. Let $a_\prec = f\{E\} \prec b$ for $\prec \in \{=, \neq\}$.

Property (a). We prove by contraposition that $J \models \tau^*(a_\neq)_I$ implies that there is no set $X \subseteq T_I$ such that $f(X \cup T_J) = b$.

Case there is a set $X \subseteq T_I$ such that $f(X \cup T_J) = b$. Let $D = \{e \in G \mid I \models B(e), H(e) \in X \cup T_J\}$. Because $T_J \subseteq T_I$ $D \not\models a_\neq$. Furthermore, we have $I \models \tau(D)^\wedge$. Observe that D contains all elements with conditions satisfied by J. Hence, we get $J \not\models \tau^*_{a_\neq}(D)^\vee$ and, in turn, $J \not\models \tau^*(a_\neq)_I$.

We prove the remaining direction, again, by contraposition.

Case $J \not\models \tau^*(a_\neq)_I$. There is a set $D \subseteq G$ such that $I \models \tau(D)^\wedge$ and $J \not\models \tau^*_{a_\neq}(D)^\vee$. Let $X = H(D)$. Because $J \not\models \tau^*_{a_\neq}(D)^\vee$, we get $f(X \cup T_J) = b$.

Property (b). This property can be shown in a similar way as the previous one. □